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fully solved.

By

M. L. KHANNA
Department of Mathematics
Meerut College, Meerut

AUTHOR OF

lgebra Made Easy ; Trigonometry Made Easy ; Differential Equations Made Easy ; Integral Calculus Made Easy ; Differential Calculus Made Easy ; Dynamics Made Easy ; Co-ordinate Geometry Made Easy ; Statics Made Easy ; Hydro-Statics Made Easy ; Theory of Equations ; Intermediate Differential Calculus etc. etc.

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PREFACE TO THE FIRST EDITION

This book has been chiefly designed to meet the requirements of the students preparing for the B. A. and B. Sc. examinations of the various Indian Universities and for those who offer Mathematics in the various competitive examinations. An attempt has been made to cover the entire course prescribed for degree classes for which it is meant.

The following are the main features of the book :—

(1) It serves the triple purpose of (i) Text book, (ii) Help book, (iii) University papers.

(2) The treatment of the theory which is a special feature of the book is in systematic and interesting manner. Even an average student shall be able to understand it independently and without any difficulty.

(3) The examples have been selected from various standard books of Indian and foreign authors on the subject. A large number of questions are such which are set in the examinations of different universities.

(4) The number of the questions is definitely very large, but they have been arranged in such a way that after doing one question of the type others follow automatically.

(5) At the end of the book Agra University papers (1937–52) have been fully solved.

Now it is useless to go into details about the fineness of the book, the students will see it for themselves, and if they really find it to be useful, I shall feel highly rewarded.

I am very much thankful to the authors and publishers of the various books I have freely consulted. My thanks are due to the publishers and printers who have made great efforts in bringing the book out.

I have to express my thanks to my worthy students Madan Lal Sharma and Vachaspati Sharma who joined me right from the start in reading proof-sheets of the book. I extend my thanks to my Post-graduate Student Ved Prakash Sharma who joined me in the mid, and gave me valuable suggestions to lay emphasis on certain points which the students really find difficult.

Meerut College,
October 1952

M. L. KHANNA

PREFACE TO THE SECOND EDITION

(1) The most important change that has been made in this edition is that references of various university questions have been given along with the questions, so that the students may attach due importance to the questions which are more favourite of the examiners.

(2) Chapters on Binomial Theorem, Exponential Theorem and Logarithmic-series have been added.

(3) All important questions have been marked by a star and students should lay more stress upon them.

(4) At various places foot-notes have been given instructing the students to leave on 1st reading certain articles and questions which are very tough and those which do not have much importance from the point of view of examination.

(5) In the end, in addition to Agra University Papers (1940–54), I have solved Rajputana University Papers (1948–54). These papers will serve as test papers for the students. After they have prepared the subject, they should open any paper and attempt it independently within an hour and a half and then compare their solutions with those given in the book. It will be a good method to know yourself and your preparation of the subject.

(6) Lists of important articles and formulae are given which the student should see a day before the examination.

(7) Certain printing mistakes which had crept in the first edition have been as far as possibte removed. In this connection I have to express my thanks to the number of students who were kind enough to point out the printing mistakes and also gave me some important suggestions.

Meerut College,
January 1955

M. L. KHANNA

PREFACE TO THE THIRD EDITION

I am extremely grateful to the professors and students of Mathematics for their kind appreciation of the book received off and on.

I have tried as far as possible to remove printing mistakes which had crept in the last edition. The university papers have been added upto date.

Meerut College,
January 1956

M. L. KHANNA

V. V. Imp. Note—Fill in the coupon given below and cut it with a blade from your book and send the same to me at the following address : Prof. M. L. Khanna, 315, Chhipi Tank, Meerut City. You will thus be entitled to receive guess papers for the University examination and other hints and guidance to prepare the subject. You will please mention on the blank back of the coupon your division in Intermediate. Let me know in which paper you are most interested and which appears to you difficult. Also please tick mark other books of mine that you are having and after you have gone through the book, let me know the misprints that you come across and also suggest any improvement that you may think fit.

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DON'T READ THIS

Those students who will simply read this book like a novel are bound to fail. You should adopt pen and paper as your companions for doing the subject. Never feel that you know a particular question simply by reading its solution and understanding it. To create in you the confidence the best method is that after you have done one chapter or more chapters, turn to the University papers given at the end and do the questions of those chapters independently and in case you are able to do them, then and only then you should feel confident of doing well in the examination. Lists of important formulae are given on page (x) for the ready reference of the students. As in other books a coupon is attached which may be sent with the coupons of other books. Students are welcome to correspond with the author on any matter regarding the book or their personal difficulty in the subject or to seek any advice, or to suggest any improvement in the book.

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Must Read This

PUBLISHER'S NOTE

We are very much proud in presenting our Mathematics series written by Prof. M. L. Khanna, who has all through his college career been a first class student, a position and merit scholarship holder. The present B. Sc. series is the result of four years of hard labour put in by the author and each book of the series serves the triple purpose of (i) **Text Book, (ii) Help Book, and (iii) University Papers, and is thus much economical. In case you want to purchase any new book of Mathematics, you are very sincerely advised in your own interest to look to the Author's book on the subject, borrow it from a friend for few days and you will find it for yourself that these books really serve the triple purpose and in case you are having it you need not supplement it by any other book. You are further advised to gain by the experience and opinions printed overleaf and expressed by various heads of the department and lecturers of mathematics from different places. In the end it gives us immense pleasure to place before you**

the fact that in various university examinations, most of the questions were set from the authors's books. Below is given the percentage of the questions of various books set in Agra and Rajputana and Punjab Universities.

1953	Algebra	100%	100%	
	Trigonometry	100%	100%	
	Differential Equations	100%	100%	
1254	Algebra	62·5%	100%	
	Trigonometry	84%	100%	
	Differential Equations	100%	100%	
	Dynamics	100%	90%	
	Integral Calculus	87·5%	100%	
	Differential Calculus	90%	90%	
19:5	Algebra	100%	100%	
	Trigonometry	100%	90%	
	Coordinate Geometry	90%	100%	
	Differential Calculus	100%	90%	
	Integral Calculus	100%	100%	
	Differential Equations	90%	100%	
	Statics	100%	90%	
	Dynamics	100%	100%	
	Hydro-Statics	100%	100%	
1956	Algebra	60%	88%	
	Trigonometry	100%	84%	100%
	Coordinate Geemetry	80%	100%	75%
	Differential Calculus	87%	100%	88%
	Integral Calculus	90%	100%	100%
	Differential Equations.	All questions were either exactly or of the type given in the book,		
	Statics	100%	80%	
	Dynamics	80%	90%	
	Hydro-Statics	100%	100%	

The publishers will feel highly obliged for any suggestion for improvement of the books.

K. N. GUPTA
General Manager
Jai Prakash Nath & Co.

LISTS OF IMPORTANT FORMULAE

List A. Partial Fractions.

Some fundamental rules of partial fractions **P. 2.**

(a) The given fraction should be a poper fraction and if it is not then divide the numerator by denominator and express as, "quotient+proper fraction."

(b) 'Make as many factors as you can of the denominator.

(c) P.F's of a fraction of the type $\frac{3x+7}{(x-3)(x-2)(x-1)}$ will be of the form.

$$\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{x-1}$$ [linear non-repeated factors]

(d) P. F's of a fraction of the type $\frac{3x+7}{(x-2)^3(x+3)}$ will be of the form.

$$\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{D}{x+3}$$

[linear repeated factors]

(c) P. F's of a fraction of the type

$$\frac{2x^2+3x+7}{(3x^2+7x+5)(x-2)(x+1)}$$

will be of the form.

$$\frac{A}{x-2}+\frac{B}{x+1}+\frac{Cx+D}{3x^2+7x+5}$$

[Non-repeated quadratic factors]

(f) P. F's of a fraction of the type $\frac{3x^2-5x+4}{(x^2-3x+2)^2(x^2+1)}$ will be of the form,

$$\frac{Ax+B}{(x^2-3x+2)}+\frac{(Cx+D)}{(x^2-3x+2)^2}+\frac{Ex+F}{x^2+1}$$

[repeated quadratic factors]

(g) **Method of Suppression.**

$$\frac{3x+7}{(x-3)(x-2)(x-1)}=\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{x-1}$$

The constant A corresponding to a factor $(x-3)$ of the denominator is obtained by putting $x=3$ everywhere in the given fraction except in $x-3$. Similarly other constants are found.

$$\therefore \quad A=\frac{3\,.\,3+7}{(3-2)(3-1)}=\frac{16}{1\,.\,2}=8 \text{ etc.}$$

This method is applied when the denominator has all its factors linear and non-repeated.

List B. Continued Fractions

(a) If $\frac{p_n}{q_n}$ be the nth Convergent corresponding to the continued fraction

$$a_1+\frac{1}{a_2+}\;\frac{1}{a_3+}\;\frac{1}{a_4+}\;\cdots\cdots\cdots\frac{1}{a_n+}\cdots\cdots\cdots \text{ then}$$

$\frac{p_n}{q_n}=\frac{a_n\,p_{n-1}+p_{n-2}}{a_n\,q_{n-1}+q_{n-2}}$ & each convergent is in its lowest term & hence $p_n=a_n\,p_{n-1}+p_{n-2}$ and $q_n=a_nq_{n-1}+q_{n-2}$.

Properties of Convergents

(i) $p_n\,q_{n-1}-q_n\,p_{n-1}=(-1)^n$ (P. 55)

(ii) $p_n\,q_{n-2}-q_n\,p_{n-2}=(-1)^{n-1}\,a_n$ (P. 57)

(iii) The odd convergents form a steadily increasing sequence of numbers and even form steadily decreasing. (P. 58)

(iv) Every odd convergent is less than every even convergent and every even convergent is greater than every odd convergent. (P. 58)

(v) The convergents of odd order are each less than the whole continued fraction and the convergents of even order are each greater than the whole continued fraction (P. 60)

(vi) The continued fraction lies between the values of any two consecutive convergents (P. 61)

(vii) Each convergent is a closer approximation to the value of the continued fraction than the preceeding convergent· (P. 62)

(viii) Limits to error made in taking any convergent for the continued fraction.

Error is greater than $\frac{1}{q_n(q_{n+1}+q_n)}$ or $\frac{1}{2(q_{n+1})^2}$ (P. 64)

Error is less than $\frac{1}{q_n q_{n+1}}$ or $\frac{1}{q_n^2 a_{n+1}}$ or $\frac{1}{q_n^2}$ (P. 65)

List C Inequalities

1. The sign of an inequality is changed if either we take reciprocal of both sides or multiply both sides by a negative quantity or transpose the quantities on either side to the other side. Thus if $a>b$, $1/a<1/b$ and $-a<-b$ and $b<a$. (P. 94)

2. The product of a number of positive quantities whose sum is constant is greatest when the quantities are equal. (P. 96)

3. Arithmetic mean of n +ive quantities is greater than their geometric mean i. e., $\left(\frac{a+b+c-......}{n}\right)>(abc-...)^{1/n}$. (P. 66)

4. $a^2+b^2+c^2>ab+bc+ca$. (P. 97)

5. The greatest value of $a^m.b^n.c^p......$ when $a+b+c$ is constant and where $m, n, p,...$ are positive integers is $m^m n^n p^p......... \left(\frac{a+b+c+......}{m+n+p+.....}\right)^{m+n+p+\cdots}$ (P. 98)

6. If a and b are unequal, then $\frac{a^m+b^m}{2}>\left(\frac{a+b}{2}\right)^m$ except when m is a +ive proper fraction. (P. 102)

7. Arithmetic mean of mth powers of n +ive quantities is greater than the arithmetic mean raised to the power m except when m lies between 0 and 1. (P. 104)

8. If both x and y are +ive proper fractions and $x>y$

then $\left(\frac{1+x}{1-x}\right)^{1/x}>\left(\frac{1+y}{1-y}\right)^{1/y}$ (P. 112)

9. If x is a +ive proper fraction, then

$(1+x)^{1+x}(1-x)^{1-x}>1$ and $(1+x)^{1-x}(1-x)^{1+x}<1$ (P. 114)

10. $a^a b^b>\left(\frac{a+b}{2}\right)^{a+b}>a^b b^a$

List D. Convergency and Divergency

1. Convergency.

A series is said to be convergent if the sum of its first n terms is such that it is always less than or does not exceed some finite quantity however great n may be. The above statement is equivalent to saying that a series will be convergent if the sum of its first n terms tends to a finite limit as n tends to infinity and is expressed as

if $\underset{n\to\infty}{\text{Lt}}\ S_n=$finite, the series is convergent (P. 153)

2. Divergency.

A series is said to be divergent if the sum of first n terms is such that it can always be made to exceed any finite quantity whatsoever you can think of by taking n sufficiently great. The above statement is equivalent to saying that a series will be divergent if the sum of its first n terms tends to either plus infinity or minus infinity as n tends to infinity and is expressed as

if $\underset{n\to\infty}{\text{Lt}}\ S_n=\infty$ or $-\infty$, the series is divergent.

3. An infinite G.P. is convergent or divergent according as its common ratio is less than unity or greater than or equal to unity. (P. 155)

4, **Some important limits.** $\underset{n\to\infty}{\text{Lt}}=\text{Lt}.$

(1) $\text{Lt}\ \frac{\log n}{n}=0$ (2) $\text{Lt}\left(1+\frac{1}{n}\right)^n=e.$

(3) $\left(1+\frac{2}{n}\right)^n=e^2$ (4) $\left(1+\frac{1}{2n}\right)^n=e^{1/2}$

(5) $\text{Lt}\frac{f(n)}{\phi(n)}=0$ or ∞ according as the degree of numerator $f(n)$ is less or greater than the degree of denominator $\phi(n)$. In case their degree be equal then the limit is the ratio of the coefficients of highest degree terms of $f(n)$ and $\phi(n)$.

5. An infinite series in which the terms are alternately posiive and negative is convergent, if each term is numerically less than the preceding term, and if the terms decrease indefinitely. **Always certify that Lt $U_n=0$.** (P. 162).

6. A series is Convergent, if from and after certain fixed term all its terms are less than the corresponding terms of a second series which is known to be convergent. (P. 166)

7. A series is divergent if from and after certain fixed term all its terms are greater than the corresponding terms of a series which is known to be divergent. (P. 167)

8. A series in which all the terms are of the same sign is divergent if each term is greater than some finite fixed quantiiy however small. (P. 169)

9. The infinite series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ i. e.

$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}\cdots\cdots+\frac{1}{n^p}\cdots\cdots$ is convergent if p be +ive and greater than 1 and divergent if p is less than or equal to 1. (P. 170)

10. If there are two series in each of which all the terms are +ive and if the ratio of the corresponding terms in the two series is always finite, the two series are either both convergent or both divergent. (P. 174)

i. e. if Lt $\frac{U_n}{V_n}$ is finite, then ΣU_n and ΣV_n are either both convergent or both divergent.

11. An infinite series of +ive terms is convergent if from and after certain fixed term the ratio of each term to the preceding term is numerically less than some quantity which is itself numerically less than unity. (P. 185)

Practical application.

Lt $\frac{U_n}{U_{n+1}} > 1$ C, Lt $\frac{U_n}{U_{n+1}} < 1$ D

Lt $\frac{U_n}{U_{n+1}} = 1$. Test fails.

12. **Cauchy's radical test or root test.** (P. 187)

ΣU_n is convergent or divergent according as Lt $(U_n)^{1/n}$ is <1 or >1 and in case the limit is 1, test fails.

13. If U_n and V_n are the general terms of two infinite series in which all the terms are +ive, then U series will be convergent when V series is given to be convergent and at the same time from and after certain fixed term.

(a) $\frac{U_{n+1}}{U_n} < \frac{V_{n+1}}{V_n}$

(b) U series will be divergent when V series is given to be divergent and at the same time from and after certain fixed term $\frac{U_{n+1}}{U_n} > \frac{V_{n+1}}{V_n}$ (P. 195)

14. **Rabee's test** : The series whose general term is convergent or divergent acccording as Lt $\left[n \left(\frac{U_n}{U_{n+1}} - 1 \right) \right]$ >1 or <1. (P. 197)

15. **Logarithmic test** : The series whose general term is U_n is convergent or divergent according as

Lt $n \log \frac{U_n}{U_{n+1}} > 1$ or <1 (P. 200)

Working Rule. (P. 203)

(a) Lt $\frac{U_n}{U_{n+8}} > 1$C and < 1D & if $= 1$ No result

Now choose either

(b) Lt $n\left[\frac{U_n}{U_{n+1}} - 1\right] > 1$C and < 1 D

Or

(c) Lt $n \log \frac{Un}{U_{n+1}} > 1$C and < 1 D,

16. **De Morgain's and Bertrand's Test.** (P. 215)

The series whose general term is U_n is convergent or divergent according as Lt $\left[\left\{n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right\} \log n\right] > 1$ or < 1 [The above test is applied when Rabee's Test fails i. e. when Lt $n\left(\frac{U_n}{U_{n+1}} - 1\right) = 1$ & remember that Lt $\frac{\log n}{n} = 0$ and it will be seen that the series comes out to be divergent] i. e. A series whose general term is U_n is divergent if $n\left(\frac{U_n}{U_{n+1}} - 1\right)$ is an algebraic function of n which tends to the limit 1 when n tends to infinity. (P. 216)

List E. Determinants

1. The value of a determinant is not altered by changing rows into columns and columns into rows. (P. 237)

2. If any two rows or columns of a determinant are interchanged, the determinant retains its absolute value but changes sign. (P. 239)

3. If any line of a determinant $\triangle$ be passed over p parallel lines the resulting determinant $\triangle' = (-1)^p \triangle$.

4. If any two rows or two columns of a determinant are identical then the determinant vanishes. (P. 241)

5. **Minor :** The determinant that is left by cencelling the row and column intersecting at a particular constituent

is called the minor of that constiuent and is denoted by the corresponding capital letter. (P. 242)

6. When the constituents of any row or any column are multiplied with their corresponding minors and summed up with alternately plus and minus sign, the result is Δ or $-\Delta$ according as the row or column whose minors are considered is made the first row, or column by even or odd crossing of rows and columns. i.e. $a_1A_1 - a_2A_2 + a_3A_3 = \Delta$ and $a_1A_2 - b_1B_2 + c_2C_2 = -\Delta$ (P. 243)

7. If the constituents of any row or column are multiplied with the minors of the constituents of some other rows or columns and summed up with alternately plus and minus sign the result is zero.

$b_1C_1 - b_2C_2 + b_3C_3 = 0$, $a_3A_1 - b_3B_1 + c_3C_1 = 0$ (P. 247)

8. **Co-factor.** The determinant that is left by cancelling the row and column intersecting at a particular constituent when that particular constituent is brought in the top left hand corner of a determinant is called its Co-factor and is denoted by the corresponding capital letter. (P. 244)

9. Co-factors are equal to minors in magnitude but are of same or of opposite sign according as the constituent whose co-factor is being calculated is brought to top left hand corner by even or odd movements of lines. (P. 245)

10. The expansion of a determinant in terms of minors and co-factors differ only in that in the former terms are alternately plus and minus while in the later all the terms are plus and that the determinant be expanded with any row or any column, the result is either zero or $+\Delta$ (and not $-\Delta$ as in minors. (P. 246)

11. If each constituent in any row or in any column be multiplied by the same factor, then the determinant is multiplied by that factor.

12. If each constituent in any row or column consists of two terms then the determinant can be expressed as the sum of two determinants. (P. 247)

13. In general if the constituents of the three columns consist of m, n, p terms respectively, the determinant can be expressed as the sum of two determinants. (P. 248)

14. If to each constituent of a line of a determinant be added or subtracted the equi-multiples of the corresponding constituents of one or more lines, the determinant remains unaltered. (P. 248)

15. If those constituents of a determinant which involve x are polynomials in x and if $\triangle$ vanishes when a be substituted for x, then $(x-a)$ is a factor of the determinant.

(P. 251)

$$\begin{vmatrix} 1 & 1 & 1 \\ a & & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

and
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

16. If $\triangle = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\triangle' = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

then $\triangle\triangle'$

$$= \begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}$$

List F. Binomial Theorem

1. $(x+a)^n = x^n + {}^nc_1 x^{n-1} a + {}^nc_2 x^{n-2}.a^2 + \ldots + {}^nc_r x^{n-r}.a^r + \ldots\ldots a^n$

2. $T_{r+1} = {}^nc_r \, . \, x^{n-r} \, . \, a^r$

3. The coefficients of terms equidistant from the beginning and the end are equal and the number of terms is one more than the index of binomial.

4. The successive coefficients in the expansion of $(1+x)^n = 1 + {}^nc_1 x + {}^nc_2 x^2 + {}^nc_3 x^3 + \ldots\ldots$ are usually denoted by c_0 (i. e. $=1$), $c_1, c_2, c_3 \ldots\ldots c_n$ and they satisfy the following properties :—

(i) $c_0 + c_1 + c_2 + c_3 + \ldots\ldots c_n = 2^n$.

(ii) $c_0 + c_2 + c_4 + \ldots\ldots = c_1 + c_3 + c_5 + \ldots\ldots = 2^{n-1}$.

5. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \ldots \ldots\ldots \infty$

and $T_{r+1} = \frac{n(n+1)(n+2) \ldots\ldots (n+r-1)}{r!} x^r$

List G. Exponential Theorem

1. $e = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{1}{n}\right)^n$ and $e^x = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{1}{n}\right)^{nx}$

2. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

3. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$

4. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$

5. $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$

e is irrational and it lies between 2 and 3.

List H. Logarithmic series

If x be less than 1, then

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$$

$$\log(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots\ldots\right)$$

$$\log(1+x)-\log(1-x)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots\right)$$

or $$\frac{1}{2}\log\frac{1+x}{1-x}=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots$$

Algebra Made Easy

CHAPTER I

PARTIAL FRACTIONS

§ 1. We are very well acquainted with the process of grouping a number of fractions connected by plus and minus signs into a single fraction. But here in this chapter we shall deal with the process of decomposing a given fraction into a number of fractions connected by plus and minus signs. This process is called **splitting into partial fractions** and the fractions thus obtained are called the **Partial Fractions of the given fraction :—**

e. g. $$\frac{2}{x-1}+\frac{3}{x-2}-\frac{4}{x-3}=\frac{x^2-10x+13}{x^3-6x^2+11x-6}$$

The fractions on the L. H. S. are called the Partial Fractions (P. F.) of the fraction on the R. H. S.

§ 2. (a) **Application :—** Suppose we are required to expand $\frac{3-2x}{1-x-6x^2}$ in powers of x, then it will be possible for us to find only the first few terms of the expansion by expanding $(1-x-6x^2)^{-1}$ binomially and multiplying them by $3-2x$. If we are further required to find out the general term, the above method will fail. But if we replace the above fraction by its partial fractions $\frac{7}{5(1-3x)}+\frac{8}{5(1+2x)}$

or
$$=\tfrac{7}{5}(1-3x)^{-1}+\tfrac{8}{5}(1+2x)^{-1}$$
$$=\tfrac{7}{5}\{1+(3x)+(3x)^2+\ldots\ldots+(3x)^r\ldots\ldots\}$$
$$+\tfrac{8}{5}\{1-(2x)+(2x)^2+\ldots\ldots+(-1)^r(2x)^r\ldots\ldots\}$$

From above general term can be easily written as

$$\left\{\tfrac{7}{5}.3^r+\tfrac{8}{5}(-1)^r.2^r\right\}x^r$$

and from the general term by giving r different values we can find the coefficients of x, x^2, x^3, and so on of any power of x we like.

(b) Again sometimes we are required to sum up a series whose general term is given. We shall split the general term into partial fractions and then find the sum of the series by method of differences :—

e. g.
$$T_n=\frac{x^{n-1}}{(1+x^n)(1+x^{n+1})}$$

$$T_n=\frac{1}{x(1-x)}\left\{\frac{1}{1+x^{n+1}}-\frac{1}{1+x^n}\right\}$$

Give to n the values 1, 2, 3, ...n.

$$T_1=\frac{1}{x(1-x)}\left\{\frac{1}{1+x^2}-\frac{1}{1+x}\right\}$$

$$T_2=\frac{1}{x(1-x)}\left\{\frac{1}{1+x^3}-\frac{1}{1+x^2}\right\}$$

$$T_n=\frac{1}{x(1-x)}\left\{\frac{1}{1+x^{n+1}}-\frac{1}{1+x^n}\right\}$$

While adding, the terms will cancel diagonally and we shall get

$$S_n=\frac{1}{x(1-x)}\left\{\frac{1}{1+x^{n+1}}-\frac{1}{x+1}\right\}$$

Proper Fraction :—

§ 3. By a proper fraction we mean a fraction of which the numerator is of lesser degree than its denominator.

§ 4. **Some fundamental rules of Partial fractions.**

(a) Before splitting a fraction into its Partial Fractions, it must be seen that the given fraction is a proper fraction. If it is not, then divide the numerator by denominator and express as "**Quotient+Proper fraction**" and then the proper fraction thus obtained will be broken into Partial fractions.

(b) The denominator shall be decomposed into as many factors as possible, e. g., a factor $(1-x^4)$ should be written as $(1-x)(1+x)(1+x^2)$.

(c) To every linear factor (non-repeated) of the form $(x-a)$ in the denominator there will be a P. F. of the form $\frac{A}{x-a}$ when A is a constant to be determined.

e. g. $\frac{3x+7}{(x-3)(x-2)(x-1)}$ will have its P. F.'s of the form

$$\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{x-1}.$$

(d) To every linear factor repeated k times of the form $(x-a)^k$ in the denominator, there will be P. F.'s of the form

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\dots+\frac{A_k}{(x-a)^k}$$

e. g. $\frac{3x+7}{(x-2)^3(x+3)}$ will have its P. F.'s of the form

$$\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{D}{(x+3)}.$$

(e) To every quadratic factor (which is non-repeated and which cannot be factorized into linear factors) of the form (ax^2+bx+c) in the denominator, there will be a P. F. of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.

e. g. $\frac{2x^2+3x+7}{(3x^2+7x+5)(x-2)(x+1)}$ will have its P. F.'s of the form $\frac{A}{x-2}+\frac{B}{x+1}+\frac{Cx+D}{3x^2+7x+5}$.

(f) To every quadratic factor which is repeated K times (but which cannot be factorised into linear factors) of the form $(ax^2+bx+c)^k$, there will be P. F.'s of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

§ 5. **General method of finding out the constants.** Express the given fraction into its partial fractions in accordance with the rules written above. Then multiply both sides by the denominator of the given fraction and you will get an identity which will be holding for all values of x. Equate the coefficients of like powers of x occurring on both sides. In this way you will obtain as many equations as the number of constants to be determined From these equations we shall find the values of these constants. The above method though quite general, but is very laborious and hence we shall give some simpler methods in some particular cases. However, if no simple method is available for certain questions, the above general method is to be applied.

Case I.

When the denominator has all the factors linear :—

Example 1. Resolve into partial fractions :—

$$\frac{2x^3-3x^2-8x-26}{2x^2-5x-12}.$$

The fraction being not a proper fraction and hence by actual division, we obtain

$$\frac{2x^3-3x^2-8x-26}{2x^2-5x-12}=x+1+\frac{9x-14}{2x^2-5x-12}$$

where $\frac{9x-14}{2x^2-5x-12}$ will be splitted into P. F.'s.

$$\frac{9x-14}{(2x+3)(x-4)}=\frac{A}{2x+3}+\frac{B}{x-4}$$

Clearing fractions by multiplying both sides with the denominator of the given fraction we obtain an identity

$$9x-14\equiv4(x-4)+B(2x+3).$$

Equating the coefficients of x and constant term in both sides of the above identity, we get

$$9=A+2B\ldots\ldots\ldots(I) \qquad A=5$$
$$-14=-4A+3B\ldots.(II) \qquad B=2$$

$$\therefore \quad \frac{9x-14}{(2x+3)(x-4)}=\frac{5}{2x+3}+\frac{2}{x-4}$$

Hence $\dfrac{2x^3-3x^2-8x-26}{2x^2-5x-12}=x+1+\dfrac{5}{2x+3}+\dfrac{2}{x-4}$

Example 2. Resolve into partial fractions :—

$$\frac{6x^2+5x-2}{2x^3-x^2-x}.$$

$$\frac{6x^2+5x-2}{x(x-1)(2x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{2x+1}$$

$$6x^2+5x-2\equiv A(x-1)(2x+1)+Bx(2x+1)+Cx(x-1)$$

Comparing the coefficients of like powers of x and constant terms on both sides of above, we get

$6=2A+2B+C\ldots\ldots.$ I Comparing x^2.

$5=-A+B-C\ldots\ldots.$II Comparing x.

$-2=-A\ldots\ldots\ldots\ldots\ldots$III Comparing constant term.

From above equations, $A=2$, $B=5$, $C=-4$.

$$\therefore \quad \text{P. F.'s are } \frac{2}{x}+\frac{3}{x-1}-\frac{4}{2x+1}.$$

Simple Method.

We have solved the above two examples by general method. Below is given a simpler method of solving the above type of examples.

Method of Suppression.

In order to find the constant corresponding to a linear factor $(x-a)$, suppress the factor $(x-a)$ in the given fraction and put $x=a$ everywhere else. The value thus obtained will be that of the constant A corresponding to $(x-a)$ in the partial fraction. Now we shall solve the above two examples by this method. In other words it means that find the values of x obtained by equating each linear factor to zero and substitute these values of x in both sides of the identity you obtain after clearing off the fractions.

Example 1. Resolve into partial fractions :—

$$\frac{9x-14}{(2x+3)(x-4)}=\frac{A}{2x+3}+\frac{B}{x-4}\text{.........(I)}$$

In order to find A put $x=-\frac{3}{2}$ in the given fraction except in the factor $(2x+3)$ which is to be suppressed.

$$A=\frac{9(-\frac{3}{2})-14}{-\frac{3}{2}-4}=5$$

Similarly $\quad B=\frac{9\cdot4-14}{(2\,4+3)}=2$ by putting $x=4$

$\therefore$ P. F.'s are $\frac{5}{2x+3}+\frac{2}{x-4}$

or $(9x-14)\equiv A(x-4)+B(2x+3)$ from (1)

Put $x=4$ and $-\frac{3}{2}$ in this identity.

Putting $x=4$, $(9.4-14)=B(2.4+3)$ $\quad\therefore\quad B=2$

Putting $x=-\frac{3}{2}$, $9(-\frac{3}{2})-14=A(-\frac{3}{2}-4)$ $\quad\therefore\quad A=5$ etc.

Example 2. Resolve into partial fractions :—

$$\frac{6x^2+5x-2}{x(x-1)(2x+1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{2x+1}$$

Put $x=0$ in the given fraction after suppressing the factor x.

$$\therefore\quad A=\frac{6.0+5.0-2}{(0-1)(0+1)}=\frac{-2}{-1}=2$$

Put $x=1$ in the given fraction after suppressing the factor $x-1$.

$$B=\frac{6.1^2+5.1-2}{1(2.1+1)}=\frac{9}{3}=3.$$ Similarly $C=-4$.

$\therefore$ Partial Fractions are $\frac{2}{x}+\frac{3}{x-1}-\frac{4}{2x+1}$

Case II. When the denominator has linear repeated factors.

Example 1. Resolve into partial fractions :—

$$\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{x+3}$$

Clearing off fractions by multiplying both sides with $(x+2)^2(x+3)$ we obtain the identity

$$(x^2+7x+11)\equiv A(x+2)(x+3)+B(x+3)+C(x+2)^2\ldots\ldots(A)$$

By general method we can compare the coefficients of x^2, x and constant and thus obtain three equations to determine the three constants. But we should first find by our simple method as many constants as there are linear factors. The rest of the constants may be found by the general method of comparing the coefficients. Here there are two linear factors and hence two constants can be easily found and the third may be found by comparing.

Put $x=-2$ in both sides of A [from $x+2=0$]

$$(4-14+11)=B(-2+3) \quad \therefore\ B=1$$

Put $x=-3$ [from $x+3=0$]

$$(9-21+11)=C(-3+2)^2 \quad \therefore\ C=-1.$$

Compare the coefficients of x^2.

$$1=A+C \quad \therefore\ A=2.$$

$\therefore$ P. F.'s are $\frac{2}{x+2}+\frac{1}{(x+2)^2}-\frac{1}{x+3}$

Example 2. Resolve into partial fractions :—

$$\frac{x^3-x^2-13x+25}{(x^2-5x+6)^2}=\frac{x^3-x^2-13x+25}{(x-2)^2(x-3)^2}$$

Let $$\frac{x^3-x^2-13x+25}{(x-2)^2(x-3)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-3)}+\frac{D}{(x-3)^2}$$

$$x^3-x^2-13x+25\equiv A(x-2)(x-3)^2+B(x-3)^2+C(x-2)^2(x-3)+D(x-2)^2$$

Put $x=2$ $\quad\therefore\quad 3=B$.

Put $x=3$ $\quad\therefore\quad 4=D$.

Compare the coefficients : Here we have to find the two constants for which we shall compare the coefficients of two terms. Always compare the coefficients of highest power of x and of constant term which are easy. If more constants are to be determined then compare the coefficients of other powers of x.

Comparing x^3, Comparing constant.

$1=A+C\ldots\ldots(1)$ $\quad 25=-18A+9B-12C+4D\ldots(2)$

Put the values of B and D in the 2nd equation.

$\therefore\quad 3=3A+2C\ldots\ldots(3)$

Solving (1) and (3), we get $A=1, C=0$.

$\therefore$ P. F.'s are $\dfrac{1}{x-2}+\dfrac{3}{(x-2)^2}+\dfrac{4}{(x-3)^2}$

Case III.

When the denominator has non-repeated quadratic factors.

Example 1. Resolve into partial fractions.

$$\frac{14x^3+14x^2-4x+3}{(3x^2-x+1)(x-1)(x+2)}=\frac{Ax+B}{3x^2-x+1}+\frac{C}{x-1}+\frac{D}{x+2}$$

$$\therefore\quad 14x^3+14x^2-4x+3=(Ax+B)(x-1)(x+2)+C(x+2)(3x^2-x+1)+D(x-1)(3x^2-x+1)$$

Here we have two linear factors and hence two constants can be determined easily.

Put $x=1$ in both sides of the identity.

$27=9C \quad\therefore\quad C=3$

Put $x=-2$ $\quad\therefore\quad -45=-45D \quad\therefore\quad D=1$

Now two more constants are to be determined for which we shall compare the coefficients of x^3 and constant term.

Comparing x^3,

$14=A+3C+3D \quad\therefore\quad A=2.$

Hence P. F.'s are $\frac{2x+1}{3x^2-x+1}+\frac{3}{x-1}+\frac{1}{x+2}$

Example 2. Resolve into partial fractions

$$\frac{4x^4+4x^3-x^2+x+1}{(x^2+x+1)(x^2-x-3)(x+1)}$$

Let the above fraction be put as

$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x-3}+\frac{E}{x+1}$$

Clearing off fractions we obtain an identity

$$\begin{aligned}4x^4+4x^3-x^2+x+1\equiv&(Ax+B)(x+1)(x^2-x-3)\\&+(Cx+D)(x+1)(x^2+x+1)\\&+E(x^2+x+1)(x^2-x-3)\ldots\ldots(A)\end{aligned}$$

Here we have one linear factor.

∴ Putting $x=-1$ in the above identity, we get

$-1=-E$ ∴ $E=1$

Now four constants are to be determined more for which we shall compare the coefficients of

x^4, x^3, x^2 and constant term.

$A+C+E=4$............(1) Put $E=1$.

$B+2C+D=4$............(2)

$-4A+2C+2D-3E=-1$.........(3)

$-3B+D-3E=1$............(4)

$A+C=3$ (5) Eliminate C between 5 & 6

$B+2C+D=4$ (6) $2A-B-D=2$ (9)

$-4A+2C+2D=2$ (7) Eliminate C between 5 & 7

$-3B+D=4$ (8) $6A-2D=4$ (10)

Eliminate A between 9 & 10

$-3B-D=2$ (11)

Solving (11) and (8) $B=-1$.

$D=1$

Putting in 10 we get $A=1$

Putting in 5 we get $C=2$

$\therefore$ $A=1, B=-1, C=2, D=1, E=1$

$\therefore$ Partial fractions are

$$\frac{x-1}{x^2+x+1}+\frac{2x+1}{x^2-x-3}+\frac{1}{x+1}$$

Note :—Students are advised to check twice the equations they obtain after comparing the coefficients.

***Alternative method of finding constants.**

After finding $E=1$, instead of comparing the coefficients, find the value of x^2 by equating to zero the quadratic factor and put the value of x^2 in both sides of A.

$x^2-x-3=0 \qquad \therefore \quad x^2=x+3$

$4(x+3)^2+4x(x+3)-(x+3)+x+1$

$\qquad =(Cx+D)(x+1)(x+3+x+1)$

$8x^2+36x+34=2(Cx+D)(x^2+3x+2)$

Again put $x^2=x+3$

or $8(x+3)+36x+34=2(Cx+D)(x+3+3x+2)$

or $44x+58=2(Cx+D)(4x+5)$

$=2\{4Cx^2+x(4D+5C)+5D\}$

$=2\{4C(x+3)+x(4D+5C)+5D\}$

$=x(8D+18C)+(24C+10D)$

Comparing the coefficients of x and constant we get

$44=8D+18C \qquad \therefore \quad C=2$

$58=24C+10D \qquad D=1$

Similarly by putting $x^2=-x-1$ and proceeding as above you will find $A=1, B=-1$,

Case IV.

When the denominator has repeated quadratic factors.

Example 1. Resolve into partial fractions.

---

*The above method is very useful & easier than the one of comparing the coefficients.

$$\frac{2x^4+2x^2+x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)^2}+\frac{Ex+F}{(x^2+1)}$$

Clear off fractions

$$2x^4+2x^2+x+1\equiv A(x^2+1)^2+(Bx+C)x+(Ex+F)x(x^2+1)$$

There is one linear factor x here. So put $x=0$ in both sides of the identity. $\therefore\ 1=A$

Compare the coefficients of x^4, x^3, x^2 and constant term. Comparing.

$$\begin{array}{ll} x^4 & \\ x^3 & \\ x^2 & \\ \text{Constant} & \end{array}\left\{\begin{array}{lll} A+E=2 & \therefore & E=1 \\ F=0 & \therefore & F=0 \\ 2A+B+E=0 & \therefore & B=-1 \\ A=1 & & \end{array}\right.$$

By comparing the constant we do not find the value of the remaining unknown constant as we have already compared the constant when we had put $x=0$ $\therefore$ comparing x, we get.

$$C+F=1 \qquad \therefore\ C=1$$

$$\text{P. F's are } \frac{1}{x}+\frac{1-x}{(1+x^2)^2}+\frac{x}{1+x^2}$$

Note :—The values of B and C can also be found by putting $x^2=-1$ as in Example 2 of Case III.

Example 2. Resolve into partial fractions :—

$$\frac{6x^4+11x^3+18x^2+14x+6}{(x+1)(x^2+x+1)^2}$$

Let it be put in the form :—

$$\frac{A}{x+1}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{(x^2+x+1)^2}$$

Clear off fractions

$$6x^4+11x^3+18x^2+14x+6=A(x^2+x+1)^2+(Bx+C)(x+1)(x^2+x+1)+(Dx+E)(x+1)$$

Put $x=-1$ in both sides of the above identity $\therefore\ A=5$

Comparing the coefficients of x^4, x^3, x^2 and constant term

$$A+B=6. \qquad \text{Put } A=5 \qquad \therefore\ B=1$$

$$
\begin{array}{lll}
2A+2B+C & =11 & \therefore C=-1 \\
3A+2B+2C+D=18 & & \therefore D=\ 3 \\
A+C+E & =6 & \therefore E=\ 2
\end{array}
$$

Hence P. F's are

$$\frac{5}{} + \frac{x-1}{} + \frac{3x+2}{}$$

Note :—The values of D and E can also be found by putting $x^2=-(x+1)$ as in Example 2 of Case III.

Example 3. Resolve into partial fractions.

$$\frac{3x^4+x^3+8x^2+x+2}{x(x^2+1)^2}$$

Let it be put in the form

$$\frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$$

Clear off fractions and we obtain an identity

$$3x^4+x^3+8x^2+x+2=A(x^2+1)^2+(Bx+C)x(x^2+1)$$
$$+(Dx+E)x \ldots\text{I}$$

Putting $x=0$, we get $2=A$

Now for the remaining four constants instead of comparing the coefficients, first put $x^2=-1$ in both sides of I.

we get

$$3-x--8+x+2=-D+Ex$$
$$-3=-D+Ex$$

Comparing the coefficients of x and constant term in above, we get, $D=3$ and $E=0$

For the rest two compare the coefficients of x^4 and x^3.

$3=A+B$ and $1=C$ $\therefore$ $B=1$ and $C=1$

Hence P. F's are $\frac{2}{x}+\frac{x+1}{x^2+1}+\frac{3x}{(x^2+1)^2}$

***Example 4. Resolve into partial fractions :—**

$$\frac{(x^2+1)^2}{x^4+x^2+1}=\frac{x^4+x^2+1+x^2}{x^4+x^2+1}$$

(Allahabad 1950)

$$=1+\frac{}{(x^2+1)^2-x^2}=1+\frac{}{(x^2+x+1)(x^2-x+1)}$$

Let $\frac{}{(x^2+x+1)(x^2-x+1)}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$

$\therefore \quad x^2=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$

Put $x^2=x-1$ in both sides (from $x^2-x+1=0$)

$$x-1=(Cx+D)(x-1+x+1)=(Cx+D)2x$$
$$=2x^2C+2Dx=2C(x-1)+2Dx$$
$$=x(2C+2D)-2C$$

Compare the coefficients of x and constant term in above

$$2C+2D=1 \quad \text{and} \quad 2C=1 \quad \therefore \quad C=\tfrac{1}{2} \quad \text{and} \quad D=0$$

Similarly putting $x^2=-x-1$ from the other quadratic factor; we get

$$-x-1=(Ax+B)(-x-1-x+1)=(Ax+B)(-2x)$$
$$=-2Ax^2-2Bx=2A(x+1)-2Bx$$
$$=(2A-2B)x+2A$$

Comparing we get

$$2A-2B=-1 \qquad \therefore \quad A=-\tfrac{1}{2}$$
$$2A=-1 \qquad \therefore \quad B=0$$

P. F.'s are $1-\frac{x}{2(x^2+x+1)}+\frac{x}{2(x^2-x+1)}$

A simpler method of breaking into partial fractions for case II.

Consider the following example.

***Ex. 1.** $\frac{x^3}{(x-1)^4(x^2-x+1)}$ **(Agra 1945)**

By our method discussed in Case II we will express the given fraction in the form

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}+\frac{Ex+F}{x^2-x+1}$$

We can find only one constant easily by putting $x=1$ in the identity which we shall get after clearing off the fraction. The rest five constants are to be determined by comparing coefficients, which will be a laborious process Below is given a simpler method.

Put $x-1=y$ i.e.

$x=y+1$ in the given fraction.

We get
$$\frac{(y+1)^3}{y^4\{(y+1)^2-(y+1)+1\}}=\frac{y^3+3y^2+3y+1}{y^4(y^2+y+1)}$$
$$\frac{1}{y^4}\left[\frac{1+3y+3y^2+y^3}{1+y+y^2}\right]\text{........ (A)}$$

$$\begin{array}{r|l|l}
1+y+y^2 & 1+3y+3y^2+y^3 & 1+2y-y^3 \\
 & \underline{1+\ y+\ y^2} & \\
 & 2y+2y^2+\ y^3 & \\
 & \underline{2y+2y^2+2y^3} & \\
 & -y^3 & \\
 & \underline{-y^3-y^4-y^5} & \\
 & y^4+y^5 &
\end{array}$$

which is of 5th degree. Hence stop here.

$\therefore$ A is now equal to

$$=\frac{1}{y^4}\left[1+2y-y^3+\frac{y^4(1+y)}{1+y+y^2}\right]$$
$$=\frac{1}{y^4}+\frac{2}{y^3}-\frac{1}{y}+\frac{1+y}{1+y+y^2}$$

Now replacing y by $x-1$, we get the required P. F.'s as

$$\frac{1}{(x-1)^4}+\frac{2}{(x-1)^3}-\frac{1}{(x-1)}+\frac{x}{x^2-x+1}$$

Procedure

(1) Put the linear repeated factor equal to y.

(2) Find the value of x in terms of y and change every term of x in terms of y.

Express the numerator and denominator of the fraction in y in ascending powers of y.

(4) Take out the factor of the form $\frac{1}{y^k}$ and then divide the numerator by denominator and continue till you get a remainder which is of **one degree less than the degree of the denominator in the given fraction,** e.g. in the above example the degree of the denominator of the given fraction is six and we continued division till we got remainder of fifth degree (i e. one less).

Ex. 2. (a) $\frac{9x^3-24x^2+48x}{(x-2)^4(x+1)}$ **(Agra 52, Banaras 30)**

Ans. $\frac{24}{(x-2)^4}+\frac{12}{(x-2)^3}+\frac{6}{(x-2)^2}+\frac{1}{(x-2)}-\frac{1}{(x+1)}$

(b) $\frac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$ **(Sagar 1948, Punjab 1935)**

Ans. $\frac{1}{(x-1)}-\frac{1}{(x+1)}+\frac{3}{(x+1)^2}-\frac{3}{(x+1)^3}+\frac{2}{(x+1)^4}$

Note :—Write (x^2-1) as $(x-1)(x+1)$

Example 3. Resolve into partial fractions.

$$\frac{x^2+1}{(x-1)^4(x^2+2)}$$ **(Agra 1930)**

Putting $x-1=y$ i.e. $x=y+1$ in the given fraction,

we get $$\frac{(y+1)^2+1}{y^4\{(y+1)^2+2\}}=\frac{y^2+2y+2}{y^4(y^2+2y+3)}$$

$$=\frac{1}{y^4}\left(\frac{2+2y+y^2}{3+2y+y^2}\right)\text{.................A}$$

Divide $2+2y+y^2$ by $3+2y+y^2$ actually and continue till remainder is of fifth degree i. e. one degree less than that of the denominator.

$$\begin{array}{r|l}
3+2y+y^2 & 2+2y+y^2 \;\big(\tfrac{2}{3}+\tfrac{2}{9}y-\tfrac{1}{27}y^2-\tfrac{4}{81}y^3 \\
 & \underline{2+\tfrac{4}{3}y+\tfrac{2}{3}y^2} \\
 & \quad \tfrac{2}{3}y+\tfrac{1}{3}y^2 \\
 & \quad \underline{\tfrac{2}{3}y+\tfrac{4}{9}y^2+\tfrac{2}{9}y^3} \\
 & \qquad -\tfrac{1}{9}y^2-\tfrac{2}{9}y^3 \\
 & \qquad \underline{-\tfrac{1}{9}y^2-\tfrac{2}{27}y^3-\tfrac{1}{27}y^4} \\
 & \qquad\quad -\tfrac{4}{27}y^3+\tfrac{1}{27}y^4 \\
 & \qquad\quad \underline{-\tfrac{4}{27}y^3-\tfrac{8}{81}y^4-\tfrac{4}{81}y^5} \\
 & \qquad\qquad\quad \tfrac{11}{81}y^4+\tfrac{4}{81}y^5
\end{array}$$

∴ A is equal to

$$\frac{1}{y^4}\left[\tfrac{2}{3}+\tfrac{2}{9}y-\tfrac{1}{27}y^2-\tfrac{4}{81}y^3+\frac{y^4(\frac{11}{81}+\frac{4}{81}y)}{3+2y+y^2}\right]$$

$$=\frac{2}{3y^4}+\frac{2}{9y^3}-\frac{1}{27y^2}-\frac{4}{81y}+\frac{4y+11}{81(3+2y+y^2)}$$

Replace y by $x-1$ ∴ Partial Fractions are

$$\frac{2}{3(x-1)^4}+\frac{2}{9(x-1)^3}-\frac{1}{27(x-1)^2}-\frac{4}{81(x-1)}+\frac{4x+7}{81(x^2+2)}$$

§ 6. How to find the General Term.

Remember the following expansions :—

$$(1-x)^{-1}=1+x+x^2+x^3+\ldots\ldots+x^r+\ldots\ldots$$

$$(1+x)^{-1}=1-x+x^2-x^3+\ldots\ldots+(-1)^r x^r+\ldots\ldots$$

∵ The expansion is $1+(-1)^1x+(-1)^2x^2+(-1)^3x^3+\ldots$

$$+(-1)^r x^r+\ldots\ldots$$

$$(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots\ldots+(r+1)x^r+\ldots\ldots$$

$$(1+x)^{-2}=1-2x+3x^2-4x^3+\ldots+(-1)^r(r+1)x^r+\ldots$$

$$(1-x)^{-3}=1+3x+6x^2+10x^3+\ldots\ldots+\frac{(r+1)(r+2)}{1.2}x^r$$

$$+\ldots\ldots\ldots$$

$$(1+x)^{-3}=1-3x+6x^2-10x^3+\ldots+(-1)^r\frac{(r+1)(r+2)}{1.2}x^r$$

$$+\ldots\ldots\ldots$$

Example 1. Find the general term of $\dfrac{2x^2+x+1}{(x+2)(3x+1)(x+3)}$

By method of suppression the partial fractions of the above fraction are easily found out to be,

$$\frac{-7}{5(x+2)}+\frac{1}{5(3x+1)}+\frac{2}{x+3}$$

In order to expand binomially each factor should be put in the form $(1\pm ax)$.

$$-7\left(1+\frac{x}{2}\right)\_\_ +\tfrac{1}{5}(1+3x)^{-1}+\tfrac{2}{3}\left(1+\frac{x}{3}\right)^{-1}$$

or $$\frac{-7}{10}\left\{1-\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^{2}-\cdots\cdots+(-1)^{r}\left(\frac{x}{2}\right)^{r}\cdots\cdots\right\}$$

$$+\tfrac{1}{5}\{1-(3x)+(3x)^{2}-\ldots+(-1)^{r}(3x)^{r}\ldots\}+\tfrac{2}{3}\left\{1-\left(\frac{x}{3}\right)\right.$$

$$\left.+\left(\frac{x}{3}\right)^{2}-\cdots\cdots+(-1)^{r}\left(\frac{x}{3}\right)^{r}\cdots\right\}$$

General term is easily found to be

$$\left\{-\tfrac{7}{10}(-1)^{r}\frac{x^{r}}{2^{r}}+\tfrac{1}{5}(-1)^{r}3^{r}x^{r}+\tfrac{2}{3}(-1)^{r}\frac{x^{r}}{3^{r}}\right\}$$

and the co-efficient of x^r is

$$(-1)^{r}\left\{\frac{-7}{10.2^{r}}+\tfrac{1}{5}3^{r}+\frac{2}{3^{r+1}}\right\}$$

*** Imp. Example 2.**

Find the coefficient of x^r in

$$\frac{7+x}{(1+x)(1+x^{2})} \qquad \textbf{(Agra 1946)}$$

The given fraction can be easily broken into partial fractions which are

$$\frac{3}{1+x}+\frac{4-3x}{(1+x^{2})}$$

Expanding binomially

$$3(1+x)^{-1}+(4-3x)(1+x^2)^{-1}$$
$$=3(1-x+x^2-x^3+\ldots\ldots+(-1)^r x^r\ldots)$$
$$+(4-3x)\{1-(x^2)+(x^2)^2-\ldots\ldots+(-1)^r(x^2)^r\ldots\}$$

We note that in the last bracket the powers of (x^2) are integers and we have to find the term of x^r, which we can get only when the power of x^2 is $\frac{r}{2}$ and the term is multiplied with 4 of the first bracket. But $\frac{r}{2}$ power will be possible only if $\frac{r}{2}$ is an integer i. e. if r is even.

Case 1. (r even)

$$3[1-x+x^2-\ldots\ldots+(-1)^r x^r\ldots\ldots]$$
$$+(4-3x)\{1-(x^2)+(x^2)^2-\ldots\ldots+(-1)^{r/2}(x^2)^{r/2}$$
$$+\ldots\ldots\ldots\}$$

The coefficient of x^r is easily collected and is equal to

$$3(-1)^r+4(-1)^{r/2}$$

But r being even $(-1)^r=1$

$\therefore$ Coefficient will be $3+4(-1)^{r/2}$

Case II. (r odd)

Now if r is odd then $r-1$ will be even and $\frac{r-1}{2}$ will be an integer and as such can be a power of x^2, giving us the term of x^{r-1} which on multiplication with $-3x$ shall be of x^r.

$$3[1-x+x^2+\ldots\ldots+(-1)^r x^r+\ldots\ldots]$$
$$+(4-3x)[1-(x^2)+(x^2)^2-\ldots\ldots+(-1)^{(r-1)/2}(x^2)^{(r-1)/2}+\ldots]$$

Coefficient of x^r is easily collected to be

$$\{3(-1)^r-3(-1)^{(r-1)/2}\}$$

But r being odd $(-1)^r=-1$

Hence the coefficient will be

$$-3[1+(-1)^{(r-1)/2}]$$

*** Ex. 3. Find the general term of**

$$\frac{2x+5}{(x-1)^3(x-3)}$$ **(Allahabad 1936, 39)**

Ans. $\frac{-11}{24}\cdot\frac{1}{3^r}+\frac{7}{2}\cdot\frac{(r+1)(r+2)}{2}+\frac{11}{4}(r+1)+\frac{11}{8}$

* § 7. Summation of series.

Example 1. Sum the series

$$\frac{1}{(1+x)(1+x^2)}+\frac{x}{(1+x^2)(1+x^3)}$$

$$+\frac{x^2}{(1+x^3)(1+x^4)}+\dots\dots\dots n \text{ terms}$$

$$T_n=\frac{x^{n-1}}{(1+x^n)(1+x^{n+1})}$$

In order to split it into P. F.'s we write

$$\left(\frac{1}{1+x^n}-\frac{1}{1+x^{n+1}}\right)=\frac{x^{n+1}-x^n}{(1+x^n)(1+x^{n+1})}$$

$$=\frac{x^{n-1}(x^2-x)}{(1+x^n)(1+x^{n+1})}=T_n(x^2-x)$$

Divide by x^2-x.

$$T_n=\frac{1}{x^2-x}\left(\frac{1}{1+x^n}-\frac{1}{1+x^{n+1}}\right)$$

Put $n=1, 2, 3\dots\dots n$

$$T_1=\frac{1}{x^2-x}\left(\frac{1}{1+x}-\frac{1}{1+x^2}\right)$$

$$T_2=\frac{1}{x^2-x}\left(\frac{1}{1+x^2}-\frac{1}{1+x^3}\right)$$

$$T_3=\frac{1}{x^2-x}\left(\frac{1}{1+x^3}-\frac{1}{1+x^4}\right)$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$T_n=\frac{1}{x^2-x}\left(\frac{1}{1+x^n}-\frac{1}{1+x^{n+1}}\right)$$

Now add these terms and we find that all the terms cancel diagonally except the first and the last.

$$\therefore \quad S_n = \frac{1}{x^2 - x}\left[\frac{1}{1+x} - \frac{1}{1+x^{n+1}}\right]$$

Note :—If $x < 1$ then $\underset{n \to \infty}{\text{Lt}} \; x^n = 0$

$$\therefore \quad S_\infty = \frac{1}{x^4 - x}\left[\frac{1}{1+x} - \frac{1}{1+0}\right]$$

$$= \frac{1}{x(x-1)}\left[\frac{1-1-x}{1+x}\right] = \frac{-x}{x(x-1)(1+x)} = \frac{-1}{x^2-1}$$

Example 2. Find the sum of the series whose nth term is

$$T_n = \frac{n}{1+n^2+n^4} = \frac{n}{(1+n^2)^2 - n^2}$$

$$T_n = \frac{n}{(1+n+n^2)(1-n+n^2)}$$

Now, T_n can be expressed into partial fractions as

$$T_n = \tfrac{1}{2}\left[\frac{1}{1-n+n^2} - \frac{1}{1+n+n^2}\right]$$

$$T_n = \tfrac{1}{2}\left[\frac{1}{1+(n-1)+(n-1)^2} - \frac{1}{1+n+n^2}\right]$$

$[\because \; 1+(n-1)+(n-1)^2 = 1+n-1+n^2-2n+1 = 1-n+n^2]$

Put $n = 1, 2, 3, \ldots\ldots n$.

$$T_1 = \tfrac{1}{2}\left[\frac{1}{1+0+0^2} - \frac{1}{1+1+1^2}\right]$$

$$T_2 = \tfrac{1}{2}\left[\frac{1}{1+1+1^2} - \frac{1}{1+2+2^2}\right]$$

$$T_3 = \tfrac{1}{2}\left[\frac{1}{1+2+2^2} - \frac{1}{1+3+3^2}\right]$$

....

$$T_n = \tfrac{1}{2}\left[\frac{1}{1+(n-1)+(n-1)^2} - \frac{1}{1+n+n^2}\right] \quad \text{Adding}$$

$$S_n=\tfrac12\left[\frac{1}{1+0+0^2}-\frac{1}{1+n+n^2}\right]=\tfrac12\left[1-\frac{1}{1+n+n^2}\right]$$

$$=\tfrac12\left[\frac{n(n+1)}{1+n+n^2}\right]$$

If we are to find S_∞ ,

$$S_\infty=\underset{n\to\infty}{\text{Lt}}\ \tfrac12\left(\frac{n^2+n}{n^2+n+1}\right)$$

$$=\underset{n\to\infty}{\text{Lt}}\ \tfrac12\left[\frac{1+\dfrac{1}{n}}{1+\dfrac{1}{n}+\dfrac{1}{n^2}}\right]=\tfrac12$$

Exercise 1

Resolve into Partial Fractions :—

1. (a) $\dfrac{x^4-5x^3+10x^2-8x-1}{(x-1)^3(x-2)}$

(Agra 1928, Lucknow 1945, Rajputana 1953)

The fraction is not a proper fraction, and hence we shall have to divide the numerator by denominator.

$$\frac{x^4-5x^3+10x^2-8x-1}{x^4-5x^3+9x^2-7x+2}=\frac{(x^4-5x^3+9x^2-7x+2)+x^2-x-3}{x^4-5x^3+9x^2-7x+2}$$

$$=1+\frac{x^2-x-3}{(x-1)^3(x-2)}$$

Breaking $\dfrac{x^2-x-3}{(x-1)^3(x-2)}$ into partial fractions, we get

$$\frac{x^4-5x^3+10x^2-8x-1}{(x-1)^3(x-2)}=1+\frac{1}{x-1}+\frac{2}{(x-1)^2}$$

$$+\frac{3}{(x-1)^3}-\frac{1}{x-2}$$

(b) $\dfrac{x^4-3x^3-3x^2+10}{(x+1)^2(x-3)}$ **(Agra 1949, Punjab 1940)**

Ans. $(x-2)+\dfrac{17}{16(x+1)}-\dfrac{11}{4(x+1)^2}-\dfrac{17}{16(x-3)}$

(c) $$\frac{1+2x}{x^2(x+2)^3(x-1)}$$ **(Punjab 1939)**

Ans. $$\frac{-3}{16x}-\frac{1}{8x^2}+\frac{11}{144(x+2)}+\frac{1}{6(x+2)^2}+\frac{1}{4(x+2)^3}+\frac{1}{9(x-1)}$$

2. (a) Resolve into partial fractions :—

$$\frac{4x^4-16x^3+17x^2-8x+7}{(x-1)(x-2)^2(x^2+1)}$$ **(Punjab 1932)**

Let the above fraction be put in the form

$$\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{Dx+E}{x^2+1}$$

Proceeding as usual we shall have to determine five constants.

For A put $x=1$, for C put $x=2$ and for D and E put $x^2=-1$.

The remaining constant B can be found by comparing the coefficient of x^4.

$\because$ P. F.'s are $$\frac{2}{x-1}+\frac{1}{x-2}-\frac{1}{(x-2)^2}+\frac{x+1}{x^2+1}$$

(b) Resolve into partial fractions :—

$$\frac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$$ **(Agra 1950)**

Ans. $$\frac{3x}{x^2+2x-5}-\frac{1}{x-3}$$

3. Resolve into partial fractions :—

$$\frac{x^3+x+1}{(x^2+1)^2(x+2)}$$ **(Punjab 1937)**

Let it be put in the form

$$\frac{A}{x+2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Clear off fractions and we obtain an identity

$$x^3+x+1 \equiv A(x^2+1)^2+(Bx+C)(x+2)(x^2+1)+(Dx+E)(x+2)$$

Put $x=-2$ in above. $A=-\frac{9}{25}$

Put $x^2=-1$ as in example 2 Page 10 giving

$$D=-\tfrac{1}{5} \text{ and } E=\tfrac{2}{5}.$$

Find B and C by comparing the coefficients of x^4 and constant term giving $B=\frac{9}{25}$ and $C=\frac{7}{25}$.

$\therefore$ P. F.'s are $-\dfrac{9}{25(x+2)}+\dfrac{9x+7}{25(x^2+1)}+\dfrac{2-x}{5(x^2+1)^2}$

4. Resolve into partial fractions :—

$$\frac{x^2-x+1}{(x-1)^2(x-2)(x^2+1)}$$ **(Agra 1941, Luck. 1942)**

Let it be put in the form

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}+\frac{Dx+E}{(x^2+1)}$$

Clear off fractions and we obtain an identity

$$x^2-x+1 \equiv A(x-1)(x-2)(x^2+1)+B(x-2)(x^2+1)$$
$$+C(x-1)^2(x^2+1)+(Dx+E)(x-2)(x-1)^2$$

Put $x=1$, $\therefore$ $B=-\frac{1}{2}$. Put $x=2$. $\therefore$ $C=\frac{3}{5}$.

Putting $x^2=-1$, we get

$$-1-x+1=(Dx+E)(x-2)(x^2-2x+1)$$

or $\quad -x=(Dx+E)(x-2)(-2x) \qquad \because x^2=-1$

$$=-2(Dx+E)(x^2-2x)$$

$$=2(Dx+E)(2x+1) \qquad \because x^2=-1$$

$$=2\{2Dx^2+x(2E+D)+E\}$$

$$=2\{x(2E+D)+(E-2D)\} \qquad \because x^2=-1$$

Compare now coefficient of x and constant term in above

$4E+2D=-1$ we get $E=-\frac{1}{5}$

$2E-4D=0$ $\quad D=\frac{1}{10}$

The remaining constant C is found by comparing the coefficient of x^4. $\because$ $A=-\frac{1}{2}$,

∴ Partial fractions are

$$-\frac{1}{2(x-1)}-\frac{1}{2(x-1)^2}+\frac{3}{5(x-2)}-\frac{x+2}{10(x^2+1)}$$

5. Resolve into partial fractions :—

$$\frac{8}{(1-x)(1+x)^2(1+x^2)}$$ (Agra 1938)

Let it be put in the form

$$\frac{A}{(1-x)}+\frac{B}{(1+x)}+\frac{C}{(1+x)^2}+\frac{Dx+E}{(1+x^2)}$$

Clear off fractions and we obtain an identity

$$8=A(1+x)^2(1+x^2)+B(1-x)(1+x)(1+x^2)$$
$$+C(1-x)(1+x^2)+(Dx+E)(1-x)(1+x)^2$$

Putting $x=1$ we get $8=8A$ ∴ $A=1$

Putting $x=-1$ we get $8=4C$ ∴ $C=2$

Now put $x^2=-1$ as in Q. 4 and we find that

$D=-2$ and $E=2$

Find B by comparing the coefficients of x^4.

$0=A-B-D$ ∴ $B=3$

Hence P. F.'s are

$$\frac{1}{1-x}+\frac{3}{1+x}+\frac{2}{(1+x)^2}+\frac{2(1-x)}{1+x^2}$$

6. (a) Resolve into partial fractions :—

$$\frac{x^6}{(x-1)(x+1)^2(x^2+1)}$$ (Punjab 1934, 1938)

The given fraction is not a proper one and hence we shall have to actually divide the numerator by denominator.

$$\frac{x^6}{(x-1)(x+1)(x^2+1)(x+1)}=\frac{x^6}{(x^2-1)(x^2+1)(x+1)}$$
$$=\frac{x^6}{(x^4-1)(x+1)}$$

$$=\frac{x^6}{x^5+x^4-x-1}=x-1+\frac{x^4+x^2-1}{x^5+x^4-x-1}$$

$$=x-1+\frac{x^4+x^2-1}{(x-1)(x+1)^2(x^2+1)}$$

Let

$$\frac{x^4+x^2-1}{(x-1)(x+1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$
$$+\frac{Dx+E}{x^2+1}$$

$$x^4+x^2-1=A(x+1)^2(x^2+1)+B(x-1)(x+1)(x^2+1)$$
$$+C(x-1)(x^2+1)+(Dx+E)(x-1)(x+1)^2$$

Put $x=1$ $\therefore$ $A=\frac{1}{8}$ Put $x=-1$ $\therefore$ $C=-\frac{1}{4}$

Put $x^2=-1$ $\therefore$ $D=-\frac{1}{4}$ and $E=\frac{1}{4}$

For B compare the coefficient of x^4, $\therefore$ $B=\frac{9}{8}$

Hence the given fraction is

$$x-1+\frac{1}{8(x-1)}+\frac{9}{8(x+1)}-\frac{1}{4(x+1)^2}-\frac{x-1}{4(x^2+1)}\ .$$

6. (b) $\dfrac{1}{(x-1)^2(x^2+1)^2}$ **(Punjab 1936)**

Ans. $\dfrac{x}{2(x^2+1)^2}+\dfrac{2x+1}{4(x^2+1)}-\dfrac{1}{2(x-1)}+\dfrac{1}{4(x-1)^2}$

***7. Resolve into partial fractions :—**

$$\frac{x^2+1}{(x-1)^4(x^3+1)}$$ **(Punjab 1933)**

Put $x-1=y$.

Changing the given fraction in terms of y, we get

$$\frac{(y+1)^2+1}{y^4\{(y+1)^3+1\}}=\frac{y^2+2y+2}{y^4(y^3+3y^2+3y+2)}$$

Writing in ascending order, we get

$$\frac{1}{y^4}\left[\frac{2+2y+y^2}{2+3y+3y^2+y^3}\right]$$

Divide and continue division till the remainder is of 6th degree because the denominator of the given fraction is of 7th degree.

∴ The given fraction is equal to

$$\frac{1}{y^4}\left[1-\tfrac{1}{2}y-\tfrac{1}{4}y^2+\tfrac{5}{8}y^3-\frac{y^4(5+13y+5y^2)}{8(2+3y+3y^2+y^3)}\right]$$

Replacing y by $x-1$, we get

$$\frac{1}{(x-1)^4}\quad\frac{1}{2(x-1)^3}-\frac{1}{4(x-1)^2}+\frac{5}{8(x-1)}-\frac{5x^2+3x-3}{8(x^3+1)}$$

Now the last fraction can be further split into partial fractions,

$$\frac{5x^2+3x-3}{x^3+1}=\frac{5x^2+3x-3}{(x+1)(x^2-x+1)}$$

which can be expressed as

$$\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

The values of A, B and C are found as usual to be as follows :—

$$A=-\tfrac{1}{3},\ B=\tfrac{16}{3}\text{ and }C=-\tfrac{8}{3}.$$

Hence the required partial fractions are

$$\frac{1}{(x-1)^4}-\frac{1}{2(x-1)^3}-\frac{1}{4(x-1)^2}+\frac{5}{8(x-1)}$$

$$-\frac{1}{8}\left[\frac{-1}{3(x+1)}+\frac{16x-8}{3(x^2-x+1)}\right]$$

or

$$\frac{1}{(x-1)^4}-\frac{1}{2(x-1)^3}-\frac{1}{4(x-1)^2}+\frac{5}{8(x-1)}+\frac{1}{24(x+1)}-\frac{2x-1}{3(x^2-x+1)}$$

8. Resolve into partial fractions :—

(a) $\dfrac{3x^3-8x^2+10}{(x-1)^4}$ **(Punjab 1941)** *(b) $\dfrac{x^4}{(x-1)^4(x+1)}$ **(Agra 1944, 1951)**

Ans. (a) $\dfrac{3}{x-1}+\dfrac{1}{(x-1)^2}-\dfrac{7}{(x-1)^3}+\dfrac{5}{(x-1)^4}$

(b) $\dfrac{1}{16(x+1)}+\dfrac{15}{16(x-1)}+\dfrac{17}{8(x-1)^2}+\dfrac{7}{4(x-1)^3}+\dfrac{1}{2(x-1)^4}$

9. Resolve into partial fractions :—

$$\frac{6+13x-3x^3}{(x-1)(x+1)^3(x+2)}$$

Proceeding as usual we have to determine five constants. Three are got by putting $x=1$, -1 and -2. The remaining two by comparing the coefficients of x^4 and constant term.

Hence P. F.'s are

$$\frac{2}{(x+1)^3}-\frac{3}{(x+1)^2}-\frac{2}{(x+1)}+\frac{2}{3(x-1)}+\frac{4}{3(x+2)}$$

10. (a) Resolve into partial fractions :—

$$\frac{x^3+5x^2+x}{(x^2+1)(x^3+1)(x+1)}=\frac{x^3+5x^2+x}{(x+1)^2(x^2+1)(x^2-x+1)}$$

Let it be put in the form

$$\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{(x^2+1)}+\frac{Ex+F}{x^2-x+1}$$

Clearing off fractions, we obtain an identity

$$x^3+5x^2+x=A(x+1)(x^2-x+1)(x^2+1)+B(x^2+1)(x^2-x+1)$$
$$+(Cx+D)(x+1)^2(x^2-x+1)+(Ex+F)(x+1)^2(x^2+1)$$

Putting $x=-1$, we get $3=6B$ $\therefore$ $B=\frac{1}{2}$

Now putting $x^2=-1$ in both sides of above and as usual we get $-5=2Cx+2D$.

Comparing the coefficients of x and constant in above, we get $C=0$ and $D=-\frac{5}{2}$.

Similarly putting $x^2=x-1$ in above, we get as usual

$$2=Ex+F$$

Comparing the coefficients of x and constant term, we get

$$E=0, \quad F=2.$$

Comparing the coefficient of x^5, we get

$$A+C+E=0 \qquad \therefore \quad A=0$$

$$A=0,\ B=\tfrac{1}{2},\ C=0,\ D=-\tfrac{5}{2},\ E=0,\ F=2.$$

$\therefore$ P. F.'s are $\dfrac{1}{2(x+1)^2}-\dfrac{5}{2(x^2+1)}+\dfrac{2}{x^2-x+1}$

(b) $\dfrac{x^3}{(x+2)^2(x^2+2)}$ **(Agra 1948)**

Ans. $\dfrac{10}{9(x+2)}-\dfrac{4}{3(x+2)^2}-\dfrac{(x+4)}{9(x^2+2)}$

***11. (a) Resolve into partial fractions :—**

$$\frac{1}{x^4+1}=\frac{1}{(x^2+1)^2-(\sqrt{2}x)^2}=\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$$

(Sagar 1949

It can be expressed in the form

$$\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{Cx+D}{x^2-\sqrt{2}x+1}$$

Clearing off the fractions, we get the identity

$1=(Ax+B)(x^2-\sqrt{2}x+1)+(Cx+D)(x^2+\sqrt{2}x+1)$

Putting $x^2=\sqrt{2}x-1$ on both sides,

$1=(Cx+D)(\sqrt{2}x-1+\sqrt{2}x+1)=(Cx+D)2\sqrt{2}x$

$=2\sqrt{2}\ Cx^2+2\sqrt{2}\ Dx$

Again putting $x^2=\sqrt{2}x-1$

$1=2\sqrt{2}C(\sqrt{2}x-1)+2\sqrt{2}\ Dx$

$=x(4C+2\sqrt{2}D)-2\sqrt{2}C$

Comparing the coefficients of x and constant term, we get

$$4C+2\sqrt{2}D=0 \qquad -2\sqrt{2}C=1$$

$$\therefore \quad C=\frac{-1}{2\sqrt{2}} \qquad \text{and} \quad D=\tfrac{1}{2}$$

Similarly putting $x^2=-\sqrt{2}x-1$, we get

$$1=(Ax+B)[-\sqrt{2}x-1-\sqrt{2}x+1]$$

$$=(Ax+B)(-2\sqrt{2}x)=-2\sqrt{2}Ax^2-2\sqrt{2}Bx$$

$$=2\sqrt{2}A(\sqrt{2}x+1)-2\sqrt{2}Bx$$

$$=x(4A-2\sqrt{2}B)+2\sqrt{2}A$$

Comparing the coefficient of x and constant term, we get

$$1=2\sqrt{2}A \qquad 4A-2\sqrt{2}B=0$$

$$\therefore \quad A=\frac{1}{2\sqrt{2}}, \quad \text{and} \quad B=\tfrac{1}{2}$$

$$\therefore \quad \text{P. F.'s are } \frac{\left(\frac{1}{2\sqrt{2}}x+\frac{1}{2}\right)}{x^2+\sqrt{2}x+1}-\frac{\left(\frac{1}{2\sqrt{2}}x-\frac{1}{2}\right)}{x^2-\sqrt{2}x+1}$$

$$=\frac{x+\sqrt{2}}{2\sqrt{2}(x^2+\sqrt{2}x+1)}-\frac{x-\sqrt{2}}{2\sqrt{2}(x^2-\sqrt{2}x+1)}$$

***(b) Resolve into partial fractions :—**

$$\frac{1}{x^4+x^2+1}$$

Proceed as in example 4 of page 13.

$$\text{P. F.'s are } \tfrac{1}{2}\frac{x+1}{x^2+x+1}-\tfrac{1}{2}\frac{x-1}{x^2-x+1}$$

(c) Resolve into partial fractions :—

$$\frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}$$

Here we need not express P. F.'s as

$$\frac{Ax+B}{x^2+a^2}+\frac{Cx+D}{x^2+b^2}+\frac{Ex+F}{x^2+c^2}$$

Even if you proceed as usual you will find that

$$A=C=E=0$$

It is better to put $x^2=t$ in the given fraction which therefore becomes

$$\frac{t}{(t+a^2)(t+b^2)(t+c^2)}$$

$$=\frac{A}{t+a^2}+\frac{B}{t+b^2}+\frac{C}{t+c^2}$$

where A by method of suppression is

$$\frac{-a^2}{(b^2-a^2)(c^2-a^2)}$$

Similarly find B and C and replace t by x^2.

Note—When all the powers of x are even, the least of them being 2, the work is simplified by the substitution $x^2=t$.

12. Find the general term in the following questions :—

(a) $$\frac{1-ax}{(1-bx)(1-cx)}$$ **(Agra 1928)**

Breaking into P.F.'s and expanding binomially, we get

G. T. is $\frac{1}{b-c}[(b-a)b^r+(c-a)c^r]x^r$

(b) $$\frac{x}{(x-a)(x-b)}$$ **(Agra 32)**

General Term

$$\frac{a^r-b^r}{a-b}\cdot\frac{x^r}{a^r . b^r}$$

*(c) $$\frac{3x^2+x-2}{(x-2)^2(1-2x)}$$ **(Agra 37, 47, Nagpur 50)**

Breaking into partial fractions, we get

$$\frac{1}{3(2x-1)}-\frac{5}{3(x-2)}-\frac{4}{(x-2)^2}$$

$$=\frac{1}{-3(1-2x)}+\frac{5}{6\left(1-\frac{x}{2}\right)}-\frac{4}{4\left(1-\frac{x}{2}\right)^2}$$

Expanding binomially we get **General Term** as

$$\left\{-\frac{1}{3}2^r+\frac{5}{6}\cdot\frac{1}{2^r}-(r+1)\cdot\frac{1}{2^r}\right\}x^r$$

$$=-x^r\left\{\frac{1}{3}2^r+\left(r+\frac{1}{6}\right)\cdot\frac{1}{2^r}\right\}.$$

Note :— The above expansion is true, provided

$$2x < 1 \quad \text{i. e.} \quad x < \tfrac{1}{2}.$$

* **(d)** $\dfrac{2x+1}{(x-1)(x^2+1)}$. Breaking into partial fractions.

(Agra '42)

$$= \frac{3}{2(x-1)} - \frac{1}{2}\,\frac{3x-1}{x^2+1},$$

$$= -\tfrac{3}{2}(1-x)^{-1} - \tfrac{1}{2}(3x-1)(1+x^2)^{-1}$$

Proceeding as in solved example 2 Page 18,

If r is even General Term is

$$\left\{ -\tfrac{3}{2} + \tfrac{1}{2}(-1)^{r/2} \right\} x^r$$

If r is odd General Term is

$$\left\{ -\tfrac{3}{2} - \tfrac{3}{2}(-1)^{\frac{r-1}{2}} \right\} x^r$$

(e) $\dfrac{1-x+2x^2}{(1-x)^3}$. Breaking into partial fractions,

$$= \frac{2}{1-x} - \frac{3}{(1-x)^2} + \frac{2}{(1-x)^3}$$

Now expanding binomially we get

$$G.T. = \left\{ 2 - 3(r+1) + 2\frac{(r+1)(r+2)}{2\,!} \right\} x^r$$

$$= (r^2+1)x^r$$

(f) $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

Breaking into partial fractions.

$$= \frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-a)(b-c)(1-bx)}$$

$$+ \frac{c^2}{(c-a)(c-b)(1-cx)}$$

Expanding binomially we get the general term as

$$=\left\{\frac{a^{r+2}}{(a-b)(a-c)}+\frac{b^{r+2}}{(b-a)(b-c)}+\frac{c^{r+2}}{(c-a)(c-b)}\right\}x^r$$

*(g) Prove that the coefficient of x^n in the expansion of $\frac{2-ax}{1-ax+bx^2}$ is $\alpha^n+\beta^n$ where α and β are the roots of the quadratic $t^2-at+b=0$.

Because α and β are the roots of $t^2-at+b=0$

$\therefore\ \alpha+\beta=a$ and $\alpha\beta=b$.

Given fraction becomes

$$\frac{2-(\alpha+\beta)x}{1-(\alpha+\beta)x+\alpha\beta x^2}=\frac{2-(\alpha+\beta)x}{(1-\alpha x)(1-\beta x)}$$

By method of suppression above is equal to

$$\frac{1}{1-\alpha x}+\frac{1}{1-\beta x}$$

and on expanding binomially, we get

$$\text{G. T.}=(\alpha^n+\beta^n)x^n$$

and hence the coefficient of x^n is $\alpha^n+\beta^n$.

(h) $\frac{3-2x^2}{(2-3x+x^2)^2}$

After breaking into partial fractions it can be put in the form

$$\frac{2}{-2\left(1-\frac{x}{2}\right)}-\frac{5}{4\left(1-\frac{x}{2}\right)^2}+\frac{2}{1-x}+\frac{1}{(1-x)^2}$$

Take the fractions in the numerator and expand binomially etc.

$$\text{G. T.}=\left\{-\frac{1}{2^r}-\frac{5}{4}\,\frac{(r+1)}{2^r}+2+(r+1)\right\}x^r$$

$$=\left(r+3-\frac{5r+9}{4.2^r}\right)x^r=\left(r+3-\frac{5r+9}{2^{r+2}}\right)x^r$$

*(i) $\dfrac{2x^2}{(x-1)^2(x^2+1)}$

P. F.'s are $\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}-\dfrac{x}{(x^2+1)}$

$$=-(1-x)^{-1}+(1-x)^{-2}-x(1+x^2)^{-1}$$

Expanding

$$=-(1+x+x^2+\dots\dots+x^r+\dots\dots\infty)$$
$$+[1+2x+3x^2+\dots\dots+(r+1)x^r+\dots\dots\infty]$$
$$-x[\{1-x^2+(x^2)^2-\dots+(-1)^r(x^2)^r+\dots\}]$$

Now if r is even. The last bracket shall give no term of x^r. Because even if we take a term $(x^2)^{r/2}$ where $r/2$ will be an integer, then it shall have to be multiplied by x and will become x^{r+1}. Hence in this case

$$\text{G. T.}=x^r(-1+r+1)=rx^r.$$

If r is odd then $\dfrac{r-1}{2}$ is an integer

$$-(1+x+x^2+\dots\dots x^r+\dots\dots\infty)+[1+2x+3x^2+\dots$$
$$\dots+(r+1)x^r+\dots\dots\infty]$$
$$-x\{1-x^2+(x^2)^2-\dots\dots+(-1)^{\frac{r-1}{2}}(x^2)^{\frac{r-1}{2}}$$
$$+\dots\dots(-1)^r(x^2)^r\dots\dots\}$$

Now $(x^2)^{\frac{r-1}{2}}$ on multiplication with x shall give the term of x^r.

$$\therefore\quad \text{G. T.}=\{-1+r+1-(-1)^{\frac{r-1}{2}}\}x^r$$
$$=\{r-(-1)^{\frac{r-1}{2}}\}x^r$$

(j) $\dfrac{x-2}{(x+2)(x-1)^2}$ (Rajputana 49)

Find as usual the partial fractions and then expand binomially.

P.F.'s are $\dfrac{-4}{9(x+2)}+\dfrac{4}{9(x-1)}-\dfrac{1}{3(x-1)^2}$

G. T.$=\{\frac{4}{9}(-\frac{1}{2})^{r+1}-\frac{1}{9}(3r+7)\}x^r$

***14. Sum the following series :— (Agra 29, 43, 53)**

(a) $\dfrac{1}{(1-x)(1-x^3)}+\dfrac{x^2}{(1-x^3)(1-x^5)}+\dfrac{x^4}{(1-x^5)(1-x^7)}+\dots$

$$T_n=\frac{x^{2n-2}}{(1-x^{2n-1})(1-x^{2n+1})}$$

Now $\dfrac{1}{1-x^{2n-1}}-\dfrac{1}{1-x^{2n+1}}=\dfrac{x^{2n-1}-x^{2n+1}}{(1-x^{2n-1})(1-x^{2n+1})}$

$$=\frac{x^{2n-2}}{(1-x^{2n-1})(1-x^{2n+1})}(x-x^3)=T_n(x-x^3)$$

Dividing by $(x-x^3)$, we get

$$T_n=\frac{1}{x-x^3}\left[\frac{1}{1-x^{2n-1}}-\frac{1}{1-x^{2n+1}}\right]$$

Giving n the values 1, 2, 3,...n and adding the terms, T_1, T_2, T_3,...T_n thus obtained, you will find that second factor of each term will cancel with the first factor of succeeding terms.

$$\therefore\ S_n=\frac{1}{x-x^3}\left[\frac{1}{1-x}-\frac{1}{1-x^{2n+1}}\right]$$

If x be < 1 then $\underset{n\to\infty}{\text{Lt}}\ x^n\to 0$

$$\therefore\quad S_\infty \text{ when } x < 1 = \frac{1}{x - x^3}\left[\frac{1}{1-x} - 1\right]$$

$$= \frac{1}{x(1-x^2)} \cdot \frac{x}{1-x} = \frac{1}{(1-x^2)(1-x)}$$

$$*\text{(b)}\quad \frac{x(1-ax)}{(1+x)(1+ax)(1+a^2x)} + \frac{ax(1-a^2x)}{(1+ax)(1+a^2x)(1+a^3x)} + \ldots$$

$$T_n = \frac{a^{n-1}x(1-a^nx)}{(1+a^{n-1}x)(1+a^nx)(1+a^{n+1}x)}$$

All the factors are linear and can be broken into partial fractions by the method of suppression.

$$\text{Let it be } \frac{A}{(1+a^{n-1}x)} + \frac{B}{(1+a^nx)} + \frac{C}{(1+a^{n+1}x)}$$

A is got by putting $x = -\frac{1}{a^{n-1}}$ everywhere in T_n except in $(1+a^{n-1}x)$

$$\therefore\quad A = \frac{-1(1+a)}{(1-a)(1-a^2)} = \frac{-1}{(1-a)^2}$$

$$\text{Similarly } B = \frac{-\frac{1}{a}(1+1)}{\left(1-\frac{1}{a}\right)(1-a)} = \frac{2}{(1-a)^2}$$

$$\text{and}\qquad C = \frac{-\frac{1}{a^2}\left(1+\frac{1}{a}\right)}{\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a}\right)} = \frac{-1}{(1-a)^2}$$

$$T_n = -\frac{1}{(1-a)^2}\left[\frac{1}{1+a^{n-1}x} - \frac{2}{1+a^nx} + \frac{1}{1+a^{n+1}x}\right]$$

$$= -\frac{1}{(1-a)^2}\left[\frac{1}{1+a^{n-1}x} - \frac{1}{1+a^nx} - \frac{1}{1+a^nx} + \frac{1}{1+a^{n+1}x}\right].$$

Put $n=1, 2, 3\ldots\ldots n$ and add the terms $T_1, T_2, T_3\ldots T_n$ thus obtained ; you will find that the last two factors in each term will cancel with the first two factors of the succeeding terms.

Hence $S_n=-\frac{1}{(1-a)^2}\left[\frac{1}{1+x}-\frac{1}{1+ax}-\frac{1}{1+a^nx}+\frac{1}{1+a^{n+1}x}\right]$

(c) $T_n=\frac{a^{n-1}}{[1+a^{n-1}x][1+a^nx]}$

Proceeding as in part *b*, we find

$$S_n=\frac{1}{1-a}\left[\frac{1}{1+x}-\frac{a^n}{1+a^nx}\right]$$

(d) $T_n=\frac{2n+1}{n^2(n+1)^2}=\frac{A}{n}+\frac{B}{n^2}+\frac{C}{(n+1)}+\frac{D}{(n+1)^2}$

By usual method $A=0=C$ and $B=1$ and $D=-1$

∴ Given fraction is $=\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]$

Putting $n=1$ 2......n and adding as before,

$$\therefore\quad S_n=\left[\frac{1}{1^2}-\frac{1}{(n+1)^2}\right]=\frac{(n+1)^2-1}{(n+1)^2}$$

$$=\frac{n^2+2n}{n^2+2n+1}$$

$$S_\infty=\underset{n\to\infty}{\text{Lt}}\frac{n^2+2n}{n^2+2n+1}=\frac{1+\frac{2}{n}}{1+\frac{2}{n}+\frac{1}{n^2}}=1$$

(e) $T_n=\frac{n+1}{n(n+2)}(-x)^{n+1}=\frac{1}{2}\left[\frac{1}{n}+\frac{1}{n+2}\right](-x)^{n+1}$

$$T_1=\frac{1}{2}\left[\frac{x^2}{1}+\frac{x^2}{3}\right]$$

$$T_2=\frac{1}{2}\left[\frac{x^3}{2}+\frac{x^3}{4}\right]$$

Note—Parts e, f, g and h may be left on first reading.

$$T_3=\tfrac{1}{2}\left[\frac{x^4}{3}+\frac{x^4}{5}\right]$$

$$T_4=-\tfrac{1}{2}\left[\frac{x^5}{4}+\frac{x^5}{6}\right]$$

... ...

We are to find the sum of infinite number of terms.

$$\therefore\quad S_\infty=\tfrac{1}{2}\left[\left(\frac{x^2}{1}-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\ldots\ldots\infty\right)\right.$$

$$\left.+\left(\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\ldots\ldots\infty\right)\right]$$

$$=\tfrac{1}{2}\left[x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots\infty\right)\right.$$

$$\left.+\frac{1}{x}\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\ldots\infty\right)\right]$$

$$=\tfrac{1}{2}\left[x\log(1+x)+\frac{1}{x}\left\{\log(1+x)-x+\frac{x^2}{2}\right\}\right].$$

$$=\tfrac{1}{2}\left[\left(x+\frac{1}{x}\right)\log(1+x)-1+\frac{x}{2}\right]$$

$$=\frac{x^2+1}{2x}\log(1+x)-\tfrac{1}{2}+\frac{x}{4}$$

(f) Sum upto n terms the series whose pth term is

$$T_p=\frac{x^p(1+x^{p+1})}{(1-x^p)(1-x^{p+1})(1-x^{p+2})}$$

Put $x^p=K$ in all those factors which are of the form $(1\pm x^a)$.

$$T_p=\frac{x^p(1+Kx)}{(1-K)(1-Kx)(1-Kx^2)}$$

Treat it as the case of non-repeated linear factors of K and x^p to be kept as it is.

$\therefore$ P. F.'s by method of suppression are

$$\frac{x^p(1+x)}{(1-k)(1-x)(1-x^2)}+\frac{\left(1+x.\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)(1-x)(1-kx)}$$

$$+\frac{x^p\left(1+\frac{1}{x}\right)}{\left(1-\frac{1}{x^2}\right)\left(1-\frac{1}{x}\right)(1-kx^2)}$$

$$=\frac{x^p}{(1-x)^2(1-k)}-\frac{2x^{p+1}}{(1-x)^2(1-kx)}+\frac{x^{p+2}}{(1-x)^2(1-kx^2)}$$

$$=\frac{1}{(1-x)^2}\left[\frac{x^p}{1-x^p}-\frac{2x^{p+1}}{1-x^{p+1}}+\frac{x^{p+2}}{1-x^{p+2}}\right]$$

$$=\frac{1}{(1-x)^2}\left[\frac{x^p}{1-x^p}-\frac{x^{p+1}}{1-x^{p+1}}-\frac{x^{p+1}}{1-x^{p+1}}+\frac{x^{p+2}}{1-x^{p+2}}\right]$$

Put $p=1, 2, 3 \ldots . n$ and add the $t_1, t_2, \ldots . t_n$ thus obtained. You will find that the last two factors of each term will cancel with the first two factors of succeeding terms.

Hence

$$S_n=\frac{1}{(1-x)^2}\left[\frac{x}{1-x}-\frac{x^2}{1-x^2}-\frac{x^{n+1}}{1-x^{n+1}}+\frac{x^{n+2}}{1-x^{n+1}}\right]$$

(g) $\dfrac{1}{(1+x)(2+x)}+\dfrac{1}{(2+x)(3+x)}+\dfrac{1}{(3+x)(4+x)}+\ldots$

$$T_n=\frac{1}{(n+x)(n+1+x)}=\left[\frac{1}{n+x}-\frac{1}{n+1+x}\right]$$

$$\therefore \quad S_n=\frac{1}{1+x}-\frac{1}{n+1+x}$$

$$S_\infty=\frac{1}{1+x}-0=\frac{1}{1+x}$$

(h) $$\frac{x+2}{(1+x)(3+2x)}+\frac{2x+2}{(3+2x)(5+4x)}+\frac{4x+2}{(5+4x)(7+8x)}+\ldots\ldots$$

$$T_n=\frac{2^{n-1}x+2}{(2n-1+2^{n-1}x)(2n+1+2^nx)}$$

By method of suppression

$$T_n=\frac{1}{(2n-1)+2^{n-1}x}-\frac{1}{(2n+1)+2^nx}$$

Now putting $n=1, 2, 3\ldots\ldots n$ and adding as before, we get

$$S_n=\frac{1}{1+x}+\frac{1}{(2n+1)+2^nx}$$

$$S_\infty=\frac{1}{1+x}-0=\frac{1}{1+x}$$

Q. 15. Prove that the sum of the homogeneous products of n dimensions which can be formed of the letters a, b, c, and their powers is

$$\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

We know

$$\frac{1}{1-ax}=(1-ax)^{-1}=(1+ax+a^2x^2+\ldots a^nx^n+\ldots\ldots)$$

$$\frac{1}{1-bx}=(1-bx)^{-1}=(1+bx+b^2x^2+\ldots b^nx^n+\ldots\ldots)$$

$$\frac{1}{1-cx}=(1-cx)^{-1}=(1+cx+c^2x^2+\ldots c^nx^n+\ldots\ldots)$$

$$\therefore\quad \frac{1}{(1-ax)(1-bx)(1-cx)}=(1+ax+a^2x^2+\ldots a^nx^n+\ldots)$$
$$(1+bx+b^2x^2+\ldots b^nx^n+\ldots)$$
$$(1+cx+c^2x^2+\ldots c^nx^n+\ldots)$$

Note—Q. 15 may be left on first reading.

Now if we multiply the three factors then it is clear that the sum of homogeneous products of a, b, c, and their powers of n dimensions will be the coefficient of x^n in the above product.

i.e. $(a^{n-4}x^{n-4})(b^2x^2)(c^2x^2)=a^{n-4}b^2c^2x^n$

$a^{n-4}b^2c^2$ is the product of n dimensions and is occurring with x^n. Similarly all other products of n dimensions will have x^n attached with them. Hence sum of the products of n dimensions=coefficient of x^n in the product of above three expansions=coefficient of x^n in

$$\frac{1}{(1-ax)(1-bx)(1-cx)} \text{ is}$$

But by Ex. 12 (f) page 31 the coefficient of x^n in

$$\frac{1}{(1-ax)(1-bx)(1-cx)}$$

$$\left\{\frac{a^{n+2}}{(a-b)(a-c)}+\frac{b^{n+2}}{(b-a)(b-c)}+\frac{c^{n+2}}{(c-a)(c-b)}\right\}$$

$$=\frac{-a^{n+2}(b-c)-b^{n+2}(c-a)-c^{n+2}(a-b)}{(a-b)(b-c)(c-a)}$$

$$=\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

$$\because \quad a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a)$$

***16. Resolve into partial fractions**

$$\frac{x^2}{(x-a)(x-b)(x-c)} \qquad \textbf{(Agra 40)}$$

and hence prove that

$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}=0$$

$\dfrac{x^2}{(x-a)(x-b)(x-c)}$ by method of suppression is equal to

$$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-a)(b-c)(x-b)}$$
$$+\frac{c^2}{(c-a)(c-b)(x-c)}$$

which are the required partial fractions.

Now putting $x=0$ in both sides we get

$$0=\frac{a^2}{(a-b)(a-c)(0-a)}+\frac{b^2}{(b-a)(b-c)(0-b)}$$

$$+\frac{c^2}{(c-a)(c-b)(0-b)}$$

$$\therefore \quad \frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}=0$$

*** 17. Resolve into partial fractions**

$$\frac{x^4}{(x-a)(x-b)(x-c)}=\frac{x^4}{x^3-x^2(a+b+c)+x(ab+bc+ca)-abc}$$

By actual division it can be put in the form

$$x+a+b+c+\frac{f(x)}{(x-a)(x-b)(x-c)}$$

where $f(x)$ is of 2nd degree and need not be found out.

$$\therefore \quad \frac{x^4}{(x-a)(x-b)(x-c)}=x+a+b+c+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$$

Multiplying both sides by $x-a$, we get

$$\frac{x^4}{(x-b)(x-c)}=(x-a)(x+a+b+c)+A+\left(\frac{B}{x-b}+\frac{C}{x-c}\right)(x-a)$$

Putting $x=a$, we find

$$\frac{a^4}{(a-b)(a-c)}=A$$

and similarly $\quad \dfrac{b^4}{(b-c)(b-a)}=B$

and $\quad \dfrac{c^4}{(c-a)(c-b)}=C$

Hence $\quad \dfrac{x^4}{(x-a)(x-b)(x-c)}=x+a+b+c$

$$+\Sigma\frac{a^4}{(a-b)(a-c)(x-a)}$$

Note :—We may remark that A and B can be written down by the method of suppression i. e. by putting $x=a$ every where except in $x-a$ even though the numerator be not of lower degree than the denominator.

Q. 18. Resolve into partial fractions.

$$\frac{(x-\alpha)(x-\beta)(x-\gamma)}{(x-a)(x-b)}$$

$$=\frac{x^3-x^2(\alpha+\beta+\gamma)+x(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma}{x^2-x(a+b)+ab}$$

$$=x-(\alpha+\beta+\gamma-a-b)+\frac{f(x)}{(x-a)(x-b)}$$

We need not calculate $f(x)$ which is of 1st degree. Quotient can be easily seen to be $x-(\alpha+\beta+\gamma-a-b)$

$$\therefore\quad \frac{(x-\alpha)(x-\beta)(x-\gamma)}{(x-a)(x-b)}=x-(\alpha+\beta+\gamma-a-b)$$

$$+\frac{A}{x-a}+\frac{B}{x-b}$$

Multiplying both sides by $(x-a)$ we get

$$\frac{(x-\alpha)(x-\beta)(x-\gamma)}{x-b}=\{x-(\alpha+\beta+\gamma-a-b)\}(x-a)$$

$$+A+B\,.\,\frac{x-a}{x-b}$$

Putting $x=a$, we get $\quad \dfrac{(a-\alpha)(a-\beta)(a-\gamma)}{a-b}=A$

Similarly $\quad \dfrac{(b-\alpha)(b-\beta)(b-\gamma)}{b-a}=B$

Note :—A could also be calculated directly by putting $x=a$ in the given fraction except in $(x-a)$. Similarly we can find B.

Q. 19. Resolve into partial fractions :—

$\dfrac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)}$ Since all the powers are even.

Putting $x^2=t$ in the given fraction, it becomes

$$\frac{(t+a^2)(t+b^2)}{(t+c^2)(t+d^2)}$$

$$=1+\frac{f(t)}{(t+c^2)(t+d^2)}$$

$$\therefore \quad 1+\frac{A}{t+c^2}+\frac{B}{t+d^2}=\frac{(t+a^2)(t+b^2)}{(t+c^2)(t+d^2)}$$

Now A and B are found as in Q. 17 and 18 and will be

$$A=\frac{(a^2-c^2)(b^2-c^2)}{(d^2-c^2)}$$

$$\text{and} \quad B=\frac{(a^2-d^2)(b^2-d^2)}{(c^2-d^2)}$$

CHAPTER II

CONTINUED FRACTIONS

§ 1. **Definitions**—An expression of the form

(a) $$a_1 + \cfrac{1}{a_2 + \cfrac{}{a_3 + \cfrac{b_4}{a_4 + \cdot}}}$$

in which each b is the numerator of a fraction whose denominator is all that follows that particular b, is called a **continued fraction.**

The continued fraction is more briefly written as

$$a_1 + \frac{b_2}{a_2 +}\;\frac{b_3}{a_3 +}\;\frac{b_4}{a_4 +}\cdots\cdots$$

(b) **Simple continued fraction—**

An expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

i.e. when all the b's are unity and every a is a $+$ive integer except that a_1 may be zero is called a **simple continued fraction.**

(c) **Terminating and non terminating, C F.**

If the number of a's in the C. F. be **finite** then it is called **terminating** C. F. and if the number of a's is **infinite** then it is called **non-terminating** C. F.

(d) Partial and Complete Quotients—

Partial Quotient.

In a simple continued fraction to which we shall confine this chapter, each of the numbers, $a_1, a_2. a_3......a_n$ are called **partial quotients.** Thus a_1 is first partial quotient, a_2 is second partial quotient, and so on.

Complete Quotient.

a_n **plus all that follows** it in the simple continued fraction is called the **nth complete quotient.** Thus in the continued fraction

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4}}}=x_1 \text{ (say)}$$

The first **complete quotient** is the fraction itself

The **second complete quotient** is $a_2+\cfrac{1}{a_3+\cfrac{1}{a_4}}=x_2$

The **third complete quotient is** $a_3+\frac{1}{a_4}=x_3$ and so on

From above we find that

$$x_1=a_1+\frac{1}{x_2}$$

$$x_2=a_2+\frac{1}{x_3}$$

i.e. $$x_n=a_n+\frac{1}{x_{n+1}}$$

where x_n stands for nth complete quotient.

Note—We will always in future mean by "quotient" as 'Partial quotient'.

§ 2. Convergents to a continued fraction.

The fraction obtained by keeping only the first, the first two, the first three......partial quotients are called the first, second, third,......convergent to the continued fraction. The nth convergent is denoted by $\frac{p_n}{q_n}$.

Thus in the simple continued fraction

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\dots\dots+\cfrac{1}{a_n}}}}$$

1st convergent $=\frac{p_1}{q_1}=\frac{a_1}{1}=\frac{a_1}{1}$

2nd convergent $=\frac{p_2}{q_2}=a_1+\frac{1}{a_2}=\frac{a_1a_2+1}{a_2}$

3rd convergent $=\frac{p_3}{q_3}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3}}=\frac{a_1a_2a_3+a_1+a_3}{a_2a_3+1}$

§ 3. To convert a given fraction into a continued fraction.

Let m/n be the given fraction, divide m by n, let a_1 be the quotient and p the remainder, then

$$\frac{m}{n}=a_1+\frac{p}{n}=a_1+\frac{1}{n/p}$$

Again divide n by p and let a_2 be the quotient and q the remainder, then $\frac{n}{p}=a_2+\frac{q}{p}=a_2+\frac{1}{p/q}$

Divide p by q and let a_3 be the quotient and r the remainder $\frac{p}{q}=a_3+\frac{r}{q}=a_3+\frac{1}{q/r}$ and so on.

Thus $$\frac{m}{n}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\dots\dots\dots\dots\dots}}}$$

In case m is less than n, then 1st quotient is zero and put $\frac{m}{n}=0+\cfrac{1}{\frac{n}{m}}$ and proceed as before.

It will be observed that the above process is the same as that of finding the G. C. M. of m and n.

Ex. 1 (a) Convert $\frac{28}{23}$ into a continued fraction and find the successive convergents.

Find the G. C. M. of 28 and 23

```
23) 28 (1
    23
    ----
     5) 23 (4
        20
        ----
        3) 5 (1
           3
           ---
           2) 3 (1
              2
              ---
              1) 2 (2
                 2
                 --
                 ×
```

The respective quotients are 1, 4, 1, 1, 2

$$\therefore \text{ C. F. is } 1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}$$

$\frac{p_1}{q_1}=1$, $\frac{p_2}{q_2}=1+\frac{1}{4}=\frac{5}{4}=1\frac{1}{4}$

$$\frac{p_3}{q_3}=1+\cfrac{1}{4+\cfrac{1}{1}}=1\tfrac{1}{5},\quad \frac{p_4}{q_4}=1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1}}}=1\tfrac{2}{9}$$

$$\frac{p_6}{q_6}=1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}=1\tfrac{5}{23}$$

Note :—The usual arithmetical process of evaluating a terminating continued fraction is to simplify the fraction step by step from the bottom to the top.

Ex. 1 (b) Convert $\frac{37}{100}$ into a continued fraction.

Ans. $\frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{3}$ **(Agra 1950)**

***Ex. 2. Convert $\sqrt{(13)}$ into a continued fraction.**

While converting irrational numbers into continued fraction the method of § 2 is not applicable. **We always take the greatest integer the number contains and proceed as shown below.**

(i) $\sqrt{(13)}$ is greater than 3 and less than 4, and hence the greatest integer next less than $\sqrt{(13)}$ is 3.

$$\sqrt{(13)}=3+[\sqrt{(13)}-3]$$

$$=3+\cfrac{1}{\cfrac{1}{[\sqrt{(13)}-3]}}=3+\cfrac{1}{\cfrac{\sqrt{(13)}+3}{13-9}}\ \text{(Note)}$$

$$=3+\cfrac{1}{\cfrac{\sqrt{(13)}+3}{4}}$$

***Successive convergents will be found more easily after reading art. 4. Note carefully the method of Example 2.**

(ii) Again the greatest integer next less than

$\frac{\sqrt{(13)}+3}{4}=1$ and hence

$$\frac{\sqrt{(13)}+3}{4}=1+\frac{\sqrt{(13)}-1}{4}=1+\cfrac{1}{\cfrac{4}{\sqrt{(13)}-1}}$$

$$=1+\cfrac{1}{\cfrac{4[\sqrt{(13)}+1]}{13-1}}=1+\cfrac{1}{\cfrac{4[\sqrt{(13)}+1]}{12}}$$

$$=1+\cfrac{1}{\cfrac{\sqrt{(13)}+1}{3}}\ldots\ldots(A)$$

(iii) Again the greatest integer next less than $\frac{\sqrt{(13)}+1}{3}$ is 1

$$\therefore\quad \frac{\sqrt{(13)}+1}{3}=1+\frac{\sqrt{(13)}-2}{3}=1+\cfrac{1}{\cfrac{3}{\sqrt{(13)}-2}}$$

$$=1+\cfrac{1}{\cfrac{3[\sqrt{(13)}+2]}{13-4}}=1+\cfrac{1}{\cfrac{\sqrt{(13)}+2}{3}}$$

(iv) Again $\frac{\sqrt{(13)}+2}{3}$ has greatest integer 1 in it.

$$=1+\frac{\sqrt{(13)}-1}{3}=1+\cfrac{1}{\cfrac{3}{\sqrt{(13)}-1}}=1+\cfrac{1}{\cfrac{3[\sqrt{(13)}+1)}{13-1}}$$

$$=1+\cfrac{1}{\cfrac{\sqrt{(13)}+1}{4}}$$

(v) Again $\frac{\sqrt{(13)}+1}{4}$ has got the greatest integer as 1

$$\therefore \quad \frac{\sqrt{(13)}+1}{4} = 1+\frac{\sqrt{(13)}-3}{4} = 1+\cfrac{1}{\cfrac{4}{\sqrt{(13)}-3}}$$

$$= 1+\cfrac{1}{\cfrac{4[\sqrt{(13)}+3]}{13-9}} = 1+\cfrac{1}{\cfrac{\sqrt{(13)}+3}{1}}$$

(vi) $\sqrt{(13)}+3$ has got the greatest integer 6.

$$\therefore \quad \sqrt{(13)}+3 \quad 6+\sqrt{(13)}-3 = 6+\cfrac{1}{\cfrac{1}{\sqrt{(13)}-3}}$$

$$= 6+\cfrac{1}{\cfrac{\sqrt{(13)}+3}{13-9}} = 6+\cfrac{1}{\cfrac{\sqrt{(13)}+3}{4}} \quad \text{......(B)}$$

Again proceed as in A for $\frac{\sqrt{(13)}+3}{4}$ and after the same number of operations you will again arrive at B and the process will continue to infinity.

$$\therefore \quad \sqrt{(13)} = 3+\underset{\times}{\frac{1}{1+}}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\underset{\times}{\frac{1}{6+}}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots\infty$$

where × × represents the beginning and the end of the cycle of partial quotients.

Ex. 3. Express ($\sqrt{3}+2$) as a simple continued fraction.

(i) Now ($\sqrt{3}+2$) is greater than 3 and less than 4 and hence the greatest integer next less than ($\sqrt{3}+2$) is 3.

$$\therefore \quad \sqrt{3}+2 = 3+\sqrt{3}-1 = 3+\cfrac{1}{\cfrac{1}{\sqrt{3}-1}}$$

$$=3+\cfrac{1}{\cfrac{\sqrt{3}+1}{3-1}}=3+\cfrac{1}{\cfrac{\sqrt{3}+1}{2}}$$

(ii) Again the integer next less than $\frac{\sqrt{3}+1}{2}$ is 1.

because $\sqrt{3}+1>2$ $\therefore$ $\frac{\sqrt{3}+1}{2}>1$

$$\therefore \quad \frac{\sqrt{3}+1}{2}=1+\frac{\sqrt{3}-1}{2}=1+\cfrac{1}{\cfrac{2}{\sqrt{3}-1}} \text{......(A)}$$

$$=1+\cfrac{1}{\cfrac{2(\sqrt{3}+1)}{3-1}}=1+\frac{1}{\sqrt{3}+1}.$$

(iii) Again the integer next less than $\sqrt{3}+1=2$

$$\therefore \quad \sqrt{3}+1=2+\sqrt{3}-1=2+\cfrac{1}{\cfrac{1}{\sqrt{3}-1}}$$

$$=2+\cfrac{1}{\cfrac{\sqrt{3}+1}{3-1}}=2+\cfrac{1}{\cfrac{\sqrt{3}+1}{2}} \text{......(B)}$$

Again we arrive at $\frac{\sqrt{3}+1}{2}$ which we have already expressed above in A. Hence from this place the cycle of partial quotients will occur.

$$\therefore \quad \sqrt{3}+2=3+\frac{1}{\underset{\times}{1+}}\;\frac{1}{\underset{\times}{2+}}\;\frac{1}{1+}\;\frac{1}{2+}\text{......}\infty$$

where × × represents the beginning and end of the cycle of partial quotients.

***§ 4. Establish the law of formation of successive convergents of simple continued fractions (Proof).**

(Agra 1932, 34, 36, 44)

Let the continued fraction be $a_1+\frac{1}{a_2+}\ \frac{1}{a_3+}\ \frac{1}{a_4+}\cdots\cdots$

then as in § 2. P. 46 $\frac{p_1}{q_1}=\frac{a_1}{1}$, $\frac{p_2}{q_2}=\frac{a_1a_2+1}{a_2}$

$a_3p_2+p_1$

$\frac{p_3}{q_3}=\frac{a_1a_2a_3+a_1+a_3}{a_2a_3+1}=\frac{a_3(a_1a_2+1)+a_1}{a_3a_2+1}$ (Note the form)

From above we find that the **numerator of 3rd convergent is formed by multiplying the 3rd quotient with the numerator of 2nd convergent and adding to it the numerator of first convergent.** The denominator of the 3rd convergent is formed in a similar way by multiplying the 3rd quotient with the denominator of 2nd convergent and adding to it the denominator of 1st convergent. Suppose that the successive convergents are formed in a similar way.

Assuming that the above law of formation holds for nth convergent then $\frac{p_n}{q_n}=\frac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}}$.........(A)

$\therefore\ p_n=a_np_{n-1}+p_{n-2}$ and $q_n=a_nq_{n-1}+q_{n-2}$.........(1)
where p_n and q_n stand for numerator and denominator of nth convergent respectively and so on.

[It will be seen that convergents are in their lowest terms (Prop. 2 Page 56).]

Now $\frac{p_n}{q_n}=a_1+\frac{1}{a_2+}\ \frac{1}{a_3+}\cdots\cdots\frac{1}{a_n}$......(1)

$$\frac{p_{n+1}}{q_{n+1}}=a_1+\frac{1}{a_2+}\ \frac{1}{a_3+}\cdots\cdots\frac{1}{a_n+}\ \frac{1}{a_{n+1}}$$

$$=a_1+\frac{1}{a_2+}\ \frac{1}{a_3+}\cdots\cdots\frac{1}{a_n+\frac{1}{a_{n+1}}}\cdots\cdots\ (2)$$

Comparing 1 and 2 we find that $\frac{p_{n+1}}{q_{n+1}}$ differs from

$\frac{p_n}{q_n}$ only in having $a_n+\frac{1}{a_{n+1}}$ for its last quotient instead of a_n.

Hence if in A instead of a_n we put $a_n+\frac{1}{a_{n+1}}$, we will get $(n+1)$th convergent in place of nth convergent.

$$\therefore \frac{p_{n+1}}{q_{n+1}}=\frac{\left(a_n+\frac{1}{a_{n+1}}\right)p_{n-1}+p_{n-2}}{\left(a_n+\frac{1}{a_{n+1}}\right)q_{n-1}+q_{n-2}}$$

$$=\frac{(a_n a_{n+1}+1)\,p_{n-1}+a_{n+1}\,p_{n-2}}{(a_n a_{n+1}+1)\,q_{n-1}+a_{n+1}\,q_{n-2}}$$

$$=\frac{a_{n+1}(a_n p_{n-1}+p_{n-2})+p_{n-1}}{a_{n+1}(a_n q_{n-1}+q_{n-2})+q_{n-1}}$$

and by our assumption,

$$=\frac{a_{n+1}\,p_n+p_{n-1}}{a_{n+1}\,q_n+q_{n-1}} \qquad \text{from (1)}$$

If we put $p_{n+1}=a_{n+1}\,p_n+p_{n-1}$

$q_{n+1}=a_{n+1}\,q_n+q_{n-1}$

We find that the numerator and denominator of the $(n+1)$th convergent follows the same law which was supposed to hold in the case of nth. But we have already seen that the law holds for third convergent, hence it holds for fourth and so on. Therefore it holds universally :—

Remember

$^*p_n=a_n\,p_{n-1}+p_{n-2}$ or $p_n-p_{n-2}=a_n\,p_{n-1}$

$^*q_n=a_n\,q_{n-1}+q_{n-2}$ or $q_n-q_{n-2}=a_n\,q_{n-1}$

Ex. 1. Prove that

$$\left(\frac{p_{n+2}}{p_n}-1\right)\left(1-\frac{p_{n-1}}{p_{n+1}}\right)=\left(\frac{q_{n+2}}{q_n}-1\right)\left(1-\frac{q_{n-1}}{q_{n+1}}\right)$$

$$\text{L. H. S.}=\frac{p_{n+2}-p_n}{p_n}\;\frac{p_{n+1}-p_{n-1}}{p_{n+1}}$$

$$=\frac{a_{n+2}\,p_{n+1}}{p_n}\quad\frac{a_{n+1}\,p_n}{p_{n+1}}=a_{n+1}\,a_{n+2}$$

Similarly R. H. S. may also be proved to be equal to

$$a_{n+1}\,a_{n+2}$$

***Ex. 2. In the continued fraction** $a_1+\dfrac{1}{a_2+}\;\dfrac{1}{a_3+}\;\dfrac{1}{a_4+}\cdots$ **prove that**

(Agra 1932, 47, 53, Lucknow 1943, Annamalai 50)

$$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\quad\frac{1}{a_{n-2}+}\cdots\cdots\frac{1}{a_1}$$

and

$$\frac{q_n}{q_{n-1}}=a_n+\frac{1}{a_{n-1}+}\quad\frac{1}{a_{n-2}+}\cdots\cdots\frac{1}{a_2}$$

$$\text{C. F.}=a_1+\frac{1}{a_2+}\quad\frac{1}{a_3+}\cdots\cdots\frac{1}{a_n+}\cdots\cdots$$

$$\frac{p_1}{q_1}=\frac{a_1}{1},\quad\frac{p_2}{q_2}=\frac{a_1a_2+1}{a_2}$$

$$\therefore\quad\frac{p_2}{p_1}=\frac{a_1a_2+1}{a_1}=a_2+\frac{1}{a_1}\quad\text{and}\quad\frac{q_2}{q_1}=a_2$$

Now

$$\frac{p_n}{p_{n-1}}=\frac{a_np_{n-1}+p_{n-2}}{p_{n-1}}=a_n+\frac{p_{n-2}}{p_{n-1}}=a_n+\frac{1}{\dfrac{p_{n-1}}{p_{n-2}}}$$

Again

$$\frac{p_{n-1}}{p_{n-2}}=\frac{a_{n-1}p_{n-2}+p_{n-3}}{p_{n-2}}=a_{n-1}+\frac{p_{n-3}}{p_{n-2}}=a_{n-1}+\frac{1}{\dfrac{p_{n-2}}{p_{n-3}}}$$

$$\therefore\quad\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\quad\frac{1}{\dfrac{p_{n-2}}{p_{n-3}}}$$

Continuing this process we get

$$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\quad\frac{1}{a_{n-2}+}\cdots\cdots\frac{1}{\dfrac{p_2}{p_1}}$$

***Ex. 2 is very important and the results should be committed to memory.**

$$= a_n + \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}+} \cdots\cdots \frac{1}{a_2+} \frac{1}{a_1}$$

Similarly $$\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1}+} \frac{1}{a_{n-2}+} \cdots\cdots \frac{1}{q_2/q_1}$$

$$= a_n + \frac{1}{a_{n-1}+} \frac{1}{a_{n-2}+} \cdots\cdots \frac{1}{a_2}$$

§ 5. Properties of convergents.

Prop. 1. If $\frac{p_n}{q_n}$ be the nth convergent of the C. F. **(Agra 33, 35, 40, 47)**

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots\cdots}}$$

then prove that $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$. **(Proof**

Note—In the above form the suffixes of p and q differ by unity and the p with greater suffix comes first.

Expanding those p and q which have greater suffix

$$p_n q_{n-1} - q_n p_{n-1} = (a_n p_{n-1} + p_{n-2}) q_{n-1} - (a_n q_{n-1} + q_{n-2}) p_{n-1}$$

By § 4 page 51,

$$= (p_{n-2} q_{n-1} - q_{n-2} p_{n-1})$$

$= (-1)(p_{n-1} q_{n-2} - q_{n-1} p_{n-2})$. Bringing in the standar form i.e. p with greater suffix first.

Again expanding those p and q which have greater suffix

$$= (-1)\{(a_{n-1} p_{n-2} + p_{n-3}) q_{n-2} - (a_{n-1} q_{n-2} + q_{n-3}) p_{n-2}\}$$

$$= (-1)(p_{n-3} q_{n-2} - q_{n-3} p_{n-2})$$

$= (-1)^2 (p_{n-2} q_{n-3} - q_{n-2} p_{n-3})$. Bringing in the standar form i.e. p with greater suffix first.

Again expanding those p and q which have greater suffi

$$= (-1)^2 \{(a_{n-2} p_{n-3} + p_{n-4}) q_{n-3} - (a_{n-2} q_{n-3} + q_{n-4}) p_{n-3}\}$$

$=(-1)^2(p_{n-4}q_{n-3}-q_{n-4}p_{n-3})$

$=(-1)^3(p_{n-3}q_{n-4}-q_{n-3}p_{n-4})$. Bringing in the standard form i.e. p with greater suffix first.

Continuing the process,

$=(-1)^{n-3}(p_3q_2-q_3p_2)$

$=(-1)^{n-2}(p_2q_1-q_2p_1)$......(1)

Now $\dfrac{p_1}{q_1}=\dfrac{a_1}{1}$, $\dfrac{p_2}{q_2}=\dfrac{a_1a_2+1}{a_2}$

$\therefore\quad p_1=a_1,\ q_1=1, \qquad p_2=a_1a_2+1,\quad q_2=a_2$

Substituting above in (i),

$=(-1)^{n-2}\{(a_1a_2+1).1-a_2a_1\}=(-1)^{n-2}(1)$

$=(-1)^{n-2}(-1)^2=(-1)^n$. Proved

Applying above formula,

$$p_7q_6-q_7p_6=(-1)^7=-1$$

$p_{11}q_{12}-q_{11}p_{12}$ (Note)

$$=-(p_{12}q_{11}-q_{12}p_{11})=-(-1)^{12}=-1$$

Prop. 5·2. Each convergent is in its lowest terms.

We know $\dfrac{p_2}{q_2}=\dfrac{a_1a_2+1}{a_2}$

and we have taken $p_2=a_1a_2+1$, $q_2=a_2$ in property no. 1.

But this is possible only when we prove that p_2 and q_2 have no common factor, and so on.

Proof—Let us suppose that p_n and q_n have a factor K, then because of this supposition $p_nq_{n-1}-q_np_{n-1}$ should also have a common factor K, or $(-1)^n$ i.e. unity should also have a common factor K, which is impossible, and hence our supposition is wrong.

$\therefore$ p_n and q_n can have no common factor.

Hence [illegible] in its lowest terms.

Pro[illegible] The difference between two successive conve[illegible] is a fraction whose numerator is unity.

[Result]

Now $\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{p_nq_{n-1}-q_np_{n-1}}{q\cdot q_{n-1}}$

$$=\frac{(-1)^n}{q_n\, q_{n-1}}$$

Prop. 5·4. For any continued fraction, Prove

$p_n\, q_{n-2}-q_n\, p_{n-2}=(-1)^{n-1}a_n$ **[Result]**

Expanding those p and q which have greater suffix

L. H. S. $=(a_np_{n-1}+p_{n-2})\, q_{n-2}-(a_nq_{n-1}+q_{n-2})p_{n-2}$

$=a_n(p_{n-1}\, q_{n-2}-q_{n-1}\, p_{n-2})=a_n(-1)^{n-1}$ by Prop. I.

Prop. 5·5. For any continued fraction, prove

$\frac{p_n}{q_n}-\frac{p_{n-2}}{q_{n-2}}=(-1)^{n-1}\frac{a_n}{q_n\, q_{n-2}}$ **[Result]**

$\frac{p_n}{q_n}-\frac{p_{n-2}}{q_{n-2}}=\frac{p_n\, q_{n-2}-q_n\, p_{n-2}}{q_n\, q_{n-2}}$

$$=\frac{(-1)^{n-1}a_n}{q_n\, q_{n-2}} \text{ by prop. 4.}$$

***Ex. 1. Prove that** $p_{n+2}q_{n-2}\sim q_{n+2}p_{n-2}=a_{n+2}a_{n+1}a_n+a_{n+2}+a_n$

$p_{n+2}q_{n-2}-q_{n+2}p_{n-2}$ on expanding those p and q which have greater suffix.

$=(a_{n+2}p_{n+1}+p_n)\, q_{n-2}-(a_{n+2}\, q_{n+1}+q_n)\, p_{n-2}$

$=a_{n+2}\{p_{n+1}\, q_{n-2}-q_{n+1}p_{n-2}\}+(p_n\, q_{n-2}-q_np_{n-2})$

$=a_{n+2}\{(a_{n+1}p_n+p_{n-1})\, q_{n-2}-(a_{n+1}\, q_n+q_{n-1})p_{n-2}\}$
$+(-1)^{n-1}a_n$ by prop. 4.

$=a_{n+2}\{a_{n+1}(p_nq_{n-2}-q_np_{n-2})+(p_{n-1}q_{n-2}-q_{n-1}p_{n-2})\}$
$+(-1)^{n-1}a_n$

$=a_{n+2}\{a_{n+1}(-1)^{n-1}a_n+(-1)^{n-1}\}+(-1)^{n-1}a_n$
(Prop. I.)

$=(-1)^{n-1}\{a_{n+2}a_{n+1}a_n+a_{n+2}+a_n\}$

$\therefore\ p_{n+2}q_{n-2}\sim q_{n+2}p_{n-2}=a_{n+2}a_{n+1}a_n$ [illegible]

***Ex. 2. Prove in a similar manner**

(Agra 38, Lucknow 41, Allahabad 42)

$p_nq_{n-4}-q_np_{n-4}=(-1)^{n-1}\{a_na_{n-1}a_{n-2}+a_n+a_{n-2}\}$

Prop. 5·6. The odd convergents form a steadily increasing sequence of numbers. **(Results)**

Consider any two successive odd convergents say

(2n−1)th and (2n+1)th

Now $$\frac{p_{2n+1}}{q_{2n+1}}-\frac{p_{2n-1}}{q_{2n-1}}=\frac{p_{2n+1}q_{2n-1}-q_{2n+1}p_{2n-1}}{q_{2n+1}q_{2n-1}}$$

$$=\frac{(-1)^{2n}a_{2n+1}}{q_{2n+1}q_{2n-1}}\text{ by Prop. 4.}$$

$$=\frac{a_{2n+1}}{q_{2n+1}q_{2n-1}}=+\text{ive or }>0$$

$$\therefore\ \frac{p_{2n+1}}{q_{2n+1}}>\frac{p_{2n-1}}{q_{2n-1}}\text{ Hence proved.}$$

Prop. 5·7. The even convergents, form a steadily decreasing sequence of numbers. **(Result)**

Consider any two successive even convergents say 2nth and (2n+2)th.

$$\frac{p_{2n+2}}{q_{2n+2}}-\frac{p_{2n}}{q_{2n}}=\frac{p_{2n+2}q_{2n}-q_{2n+2}p_{2n}}{q_{2n}q_{2n+2}}$$

$$=\frac{(-1)^{2n+1}a_{2n+2}}{q_{2n}\,q_{2n+2}}\text{ by prop. 4.}$$

$$=-\frac{a_{2n+2}}{q_{2n}q_{2n+2}}=-\text{ive or }<0$$

$$\therefore\ \frac{p_{2n+2}}{q_{2n+2}}<\frac{p_{2n}}{q_{2n}}\text{ Hence proved.}$$

Prop. 5·8 Every odd convergent is less than every even convergent and every even convergent is greater than every odd convergent. **[Proof and result]**

$$\frac{p_{2n+1}}{q_{2n+1}}-\frac{p_{2n}}{q_{2n}}=\frac{p_{2n+1}q_{2n}-q_{2n+1}p_{2n}}{q_{2n}q_{2n+1}}$$

$$=\frac{(-1)^{2n+1}}{q_{2n}q_{2n+1}}\text{ by prop. 1}=-\text{ive i. e. }<0$$

$\therefore \dfrac{p_{2n+1}}{q_{2n+1}} < \dfrac{p_{2n}}{q_{2n}}$............(A) ✓

But by property No. 7 we know that even convergents form a steadily decreasing sequence.

i. e. $\dfrac{p_2}{q_2} > \dfrac{p_4}{q_4} > \dfrac{p_6}{q_6} \ldots\ldots > \dfrac{p_{2n}}{q_{2n}}$ ✓......(B)

Putting $n=1$ in A, we get $\dfrac{p_3}{q_3} < \dfrac{p_2}{q_2}$

Putting $n=3$ in A, we get $\dfrac{p_7}{q_7} < \dfrac{p_6}{q_6}$, and by B, $< \dfrac{p_4}{q_4} < \dfrac{p_2}{q_2}$

Hence by A and B **every odd convergent is less than every even convergent that precedes it.**

i.e. $\dfrac{p_9}{q_9} < \dfrac{p_8}{q_8} < \dfrac{p_6}{q_6}$ etc.........(C)

Again we know by A that

$\dfrac{p_{2n}}{q_{2n}} > \dfrac{p_{2n+1}}{q_{2n+1}}$......($A_1$) (from A)

But by property No. 6 we know that odd convergents form a steadily increasing sequence.

i.e. $\dfrac{p_1}{q_1} < \dfrac{p_3}{q_3} < \dfrac{p_5}{q_5} \ldots\ldots < \dfrac{p_{2n+1}}{q_{2n+1}}$.........($B_1$)

Putting $n=1$ in A_1, we get $\dfrac{p_2}{q_2} > \dfrac{p_3}{q_3}$ and by B_1, $> \dfrac{p_1}{q_1}$

Putting $n=3$ in A_1 $\dfrac{p_6}{q_6} > \dfrac{p_7}{q_7}$ and by B_1, $> \dfrac{p_5}{q_5} > \dfrac{p_3}{q_3} > \dfrac{p_1}{q_1}$

Hence by A_1 and B_1 we prove that **every even convergent is greater than every odd convergent that precedes it.**

i.e. $\dfrac{p_{10}}{q_{10}} > \dfrac{p_9}{q_9} > \dfrac{p_7}{q_7}$ etc......(C_1)

Now from (C) every odd convergent is less than every even convergent that precedes it.

i. e. $\dfrac{p_9}{q_9} < \dfrac{p_8}{q_8}$ etc. & from (C_1) $\dfrac{p_{10}}{q_{10}} > \dfrac{p_9}{q_9}$ i.e. $\dfrac{p_9}{q_9} < \dfrac{p_{10}}{q_{10}}$ i. e. 9th convergent is less than both 8th and 10th. Hence we can say that every odd convergent is less than every even convergent. Arguing as above we can say that every even convergent is greater than every odd convergent.

***Prop. 5·9. The convergents of odd order are each less than the whole continued fraction and the convergents of even order are each greater than the whole continued fraction. (Proof)**

Proof—Let the C. F.

$$X=a_1+\frac{1}{a_2+}\;\frac{1}{a_3+}\cdots\cdots\frac{1}{a_{n+1}+}\;\frac{1}{a_{n+2}+}\cdots\cdots$$

then $$\frac{p_{n+1}}{q_{n+1}}=a_1+\frac{1}{a_2+}\;\frac{1}{a_3+}\cdots\cdots\frac{1}{a_{n+1}}$$

Where a_{n+1} is the $(n+1)$th partial quotient

and $a_{n+1}+\frac{1}{a_{n+2}+}\;\frac{1}{a_{n+3}+}\ldots$ is $(n+1)$th complete quotient k.

We know $\frac{p_{n+1}}{q_{n+1}}=\frac{a_{n+1}\,p_n+p_{n-1}}{a_{n+1}\,q_n+q_{n-1}}$

If in the above we replace the partial quotient a_{n+1} by the complete quotient k then $\frac{p_{n+1}}{q_{n+1}}$ shall become X i. e. the whole continued fraction.

$$\therefore\quad X=\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}}$$

$$\therefore\quad X-\frac{p_n}{q_n}=\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}}-\frac{p_n}{q_n}$$

$$=\frac{p_{n-1}q_n-q_{n-1}p_n}{q_n(kq_n+q_{n-1})}$$

$$=-\frac{(p_nq_{n-1}-q_np_{n-1})}{q_n(kq_n+q_{n-1})}$$

$$=-\frac{(-1)^n}{q_n(kq_n+q_{n-1})}$$ (by prop. No. 1) and this is +ive or −ive according as n is odd or even.

When n is odd then $X - \frac{p_n}{q_n}$ is +ive or > 0.

$$\therefore \quad X > \frac{p_n}{q_n} \text{ or } \frac{p_n}{q_n} < X$$

which proves that the convergents of odd order are each less than the continued fraction.

When n is even then $X - \frac{p_n}{q_n} = -$ive or < 0

$$\therefore \quad X < \frac{p_n}{q_n} \text{ or } \frac{p_n}{q_n} > X$$

which proves that the convergents of even order are each greater than the continued fraction.

Note—Combining the result of Prop. 9 with those of Prop. 6 and 7, we get the following :—

- *(a) **The convergents of an odd order continually increase but are always less than the continued fraction** **(Prop. 6, 9)**
- *(b) **The convergents of even order continually decrease but are always greater than the continued fraction.** **(Prop. 7, 9)**

Prop. 5·10. The continued fraction lies between the values of any two consecutive convergents.

(Result)

Out of the two consecutive convergents, one of them must be odd and other even and by prop. 9.

We know odd convergent $< X$

and even convergent $> X$

i.e. Odd convergent $< X <$ even convergent.

Hence proved.

***Prop. 5·11. Each convergent is a closer approxi-**

***Article for proof can be put in the form (a) and (b) given above. § 5·11 is important for proof.**

mation to the value of the continued fraction than the preceding convergent. **(Agra 1942)**

Just like in prop. 9 if X is the value of C. F. and k is the $(n+1)$th complete quotient then

$$X-\frac{p_n}{q_n}=-\frac{(-1)^n}{q_n(kq_n+q_{n-1})}\dots\dots\dots A$$

$$X-\frac{p_{n-1}}{q_{n-1}}=\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}}-\frac{p_{n-1}}{q_{n-1}}$$

$$=\frac{k(p_nq_{n-1}-q_np_{n-1})}{q_{n-1}(kq_n+q_{n-1})}=\frac{k(-1)^n}{q_{n-1}(kq_n+q_{n-1})}\dots\dots(B)$$

Dividing A by B, we get

$$\frac{X-\frac{p_n}{q_n}}{X-\frac{p_{n-1}}{q_{n-1}}}=-\frac{q_{n-1}}{kq_n}\dots\dots\dots(C)$$

Now k, q_n, q_{n-1} are all integers and also

$q_n > q_{n-1}$ and $k > 1$ $\therefore$ $\frac{q_{n-1}}{kq_n}$ is numerically less than unity because denominator is > numerator.

$\therefore$ $X-\frac{p_n}{q_n}$ is numerically less than $X-\frac{p_{n-1}}{q_{n-1}}$

In other words $\frac{p_n}{q_n}$ differs from X by less than as much as $\frac{p_{n-1}}{q_{n-1}}$ differs from X.

Hence nth convergent is nearer in value to the whole continued fraction than the preceding (n−1)th convergent.

***Prop. 5·12. To find the limits to error made in taking any convergent for the continued fraction.**

(Proof)

Let $\frac{p_n}{q_n}$ be the nth convergent and X the value of the continued fraction then the numerical error in taking

§ 5-12 is very important for proof. Any of its result A, B, C, D or E can be asked.

$\frac{p_n}{q_n}$ instead of X is $X \sim \frac{p_n}{q_n}$

$\left\{ X \sim \frac{p_n}{q_n} \text{ denotes both } X - \frac{p_n}{q_n} \text{ and } \frac{p_n}{q_n} - X \right\}$

Let $X = a_1 + \frac{1}{a_2+} \frac{1}{a_3+} \cdots \cdots \frac{1}{a_n+} \frac{1}{a_{n+1}+} \frac{1}{a_{n+2}+} \cdots \cdots$

$$\frac{p_{n+2}}{q_{n+2}} = a_1 + \frac{1}{a_2+} \frac{1}{a_3+} \cdots \cdots \frac{1}{a_{n+1}} \frac{1}{a_{n+2}}$$

$$= \frac{a_{n+2}p_{n+1} + p_n}{a_{n+2}q_{n+1} + q_n}$$

Let $k = (n+2)$th complete quotient

$$= a_{n+2} + \frac{1}{a_{n+3}+} \frac{1}{a_{n+4}+} \cdots \cdots$$

If in $(n+2)$th convergent we put instead of partial quotient (a_{n+2}), the complete quotient k, then it becomes the continued fraction itself.

$\therefore \quad X = \frac{kp_{n+1} + p_n}{kq_{n+1} + q_n}$

$\therefore$ The error is

$$X \sim \frac{p_n}{q_n} = \frac{kp_{n+1} + p_n}{kq_{n+1} + q_n} \sim \frac{p_n}{q_n}$$

$$= \frac{k(p_{n+1}q_n \sim q_{n+1}p_n)}{q_n(kq_{n+1} + q_n)} = \frac{k}{q_n(kq_{n+1} + q_n)}$$

by Prop. I.

$$\therefore \quad \text{Error} = \frac{1}{q_n\left(q_{n+1} + \frac{q_n}{k}\right)} \ldots (A)$$

Rough lower limit. 1st form.

Now $k > 1 \;\therefore\; \frac{q_n}{k} < q_n \;\therefore\; q_n\left(q_{n+1} + \frac{q_n}{k}\right) < q_n(q_{n+1} + q_n)$

Taking reciprocal and changing the sign of inequality,

we get $$\frac{1}{q_n\left(q_{n+1}+\frac{q_n}{k}\right)} > \frac{1}{q_n(q_{n+1}+q_n)}$$

and $\therefore$ by A, error is $> \frac{1}{q_n(q_{n+1}+q_n)}$......(B)

which gives the lower limit to the error

Second form for lower limit.

We know that $q_n < q_{n+1}$

$\therefore \quad q_n(q_{n+1}+q_n) < q_{n+1}(q_{n+1}+q_{n+1})$ i.e. $< 2(q_{n+1})^2$

$$\frac{1}{q_n(q_{n+1}+q_n)} > \frac{1}{2(q_{n+1})^2}$$

$\therefore$ From B error is $> \frac{1}{q_n(q_{n+1}+q_n)} > \frac{1}{2(q_{n+1})^2}$

which may be taken as the rough lower limit.

Rough upper limit—1st form.

Again $\frac{q_n}{k} = +$ive i.e. > 0

$$\therefore \quad q_n\left(q_{n+1}+\frac{q_n}{k}\right) > q_n(q_{n+1})$$

Taking reciprocal and changing the sign of inequality.

$$\frac{1}{{}_n\left(q_{n+1}+\frac{q_n}{k}\right)} < \frac{1}{q_n(q_{n+1})}$$

$\therefore$ by A, error is $< \frac{1}{q_n(q_{n+1})}$.........C

which gives the upper limit to the error.

Second form for upper limit.

But $q_{n-1} = +$ive > 0

$\therefore \quad q_n q_{n+1} = q_n(a_{n+1}q_n + q_{n-1}) > q_n(a_{n+1}q_n)$

Taking reciprocal and changing the sign of inequality

$$\frac{1}{q_n q_{n+1}} = \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} < \frac{1}{q_n(a_{n+1}q_n)}$$

Hence from (C) error is less than $\frac{1}{q_n^2 a_{n+1}}$......(D)

This can be taken as the rough upper limit and this error is small when a_{n+1} is large and hence nth convergent will be very closely approximating to X : **Therefore if any of the quotients is very large the convergent just preceding it gives a fairly close approximation to the continued fraction. (Remember)**

Third form for upper limit.

Again from D we find that the error is less than

$$\frac{1}{a_{n+1}q_n^2}$$

Now a_{n+1} is $(n+1)$th partial quotient and hence is +ive and must be integer which is either equal or greater than 1.

$$a_{n+1} \geqslant 1$$

$$\therefore \quad q_n^2 a_{n+1} \geqslant q_n^2$$

Taking reciprocal and changing the sign of inequality we get

$$\frac{1}{q_n^2 a_{n+1}} \leqslant \frac{1}{q_n^2}$$

Hence from D error is less than $\frac{1}{q_n^2}$.........(E)

This can be taken as the rough upper limit.

***Ex. 1.** Find a series of fractions approximating to e and find that fraction which represents e very closely, $e = 2{\cdot}71828$.

Now $e = 2{\cdot}71828 = \frac{271828}{100000}$

Converting this into a continued fraction by the method of finding G. C. M. we find that the quotients are

2, 1, 2, 1, 1, 4, 1, 1, 6, 10, 1, 1, 2

$$\therefore \quad \text{C. F.}=2+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots\cdots$$

The successive convergents are

$$\frac{p_1}{q_1}=\frac{2}{1},\ \frac{p_2}{q_2}=2+\frac{1}{1}=\frac{3}{1},\ \frac{p_3}{q_3}=\frac{a_3p_2+p_1}{a_3q_2+q_1}=\frac{2.3+2}{2.1+1}=\frac{8}{3}$$

Similarly $\frac{p_4}{q_4}=\frac{11}{4}$, $\frac{p_5}{q_5}=\frac{19}{7}$, $\frac{p_6}{q_6}=\frac{87}{32}$

$\frac{p_7}{q_7}=\frac{106}{39}$, $\frac{p_8}{q_8}=\frac{193}{71}$, $\frac{p_9}{q_9}=\frac{1264}{465}$, $\frac{p_{10}}{q_{10}}=\frac{12833}{4721}$

Now by prop. (11) we know that each convergent is a closer approximation to the value of C.F. than the preceding one.

Hence all these convergents form a series of fractions approximating to *e*.

Again by property 12. D we find that the convergent which preceds the large quotient 10 is a very near approximation, the error being less than

$\frac{1}{q_n{}^2 \cdot a_{n+1}}$. Now 10 being 10th quotient

$\therefore$ 9th convergent i.e. $\frac{1264}{465}$ is a closer approximation to C.F.

The error is $< \frac{1}{(465)(465).(10)}$

***Ex. 2. Find a series of fractions approximating to**

3·14159

In the process of finding the G. C. M. of 314159 and 100000 the successive quotients are 3, 7, 15, 1, 25, 1, 7, 4.

$$\therefore \quad \text{C. F. is } 3+\frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{25+}\,\frac{1}{1+}\,\frac{1}{7+}\,\frac{1}{4+}$$

Successive convergents are

$$\frac{p_1}{q_1}=\frac{3}{1},\quad \frac{p_2}{q_2}=3+\frac{1}{7}=\frac{22}{7},\quad \frac{p_3}{q_3}=\frac{a_3p_2+p_1}{a_3q_2+q_1}=\frac{333}{106}$$

$$\frac{p_4}{q_4}=\frac{a_4p_3+p_2}{a_4q_3+q_2}=\frac{355}{113}\ \text{and so on.}$$

All these convergents by prop. (11) approximate to 3.14159

Again by prop. 12. D. the convergent which precedes the large quotient 25 is a very near approximation to 3.14159 and the error is less than $\frac{1}{a_{n+1}q_n^2}$ i e. $<\frac{1}{25.(113)^2}$

∴ $<\frac{1}{25.(100)^2}$ as well. Or $<$ ·000004

***§ 6. (a) If we are given two +ive integers A and B Prime to one another, we can always find two others, P and Q such that QA − PB = ±1.**

Express $\frac{A}{B}$ as a simple continued fraction whose quotients are say n, then $\frac{A}{B}=\frac{p_n}{q_n}$ i. e. $p_n=A$ and $q_n=B$.

and let $\frac{P}{Q}$ be the last but one i.e. $(n-1)$th convergent.

∴ $\frac{P}{Q}=\frac{p_{n-1}}{q_{n-1}}$ i. e. $p_{n-1}=P$ and $q_{n-1}=Q$.

then $QA-PB=p_nq_{n-1}-q_np_{n-1}=(-1)^n=\pm1$ by prop. I.

(b) Similarly if we are given polynomials A and B having no common factor we can find two other polynomials, P and Q such that QA − PB = 1.

***Ex. 1. Find two +ive integers P and Q such that**

$$37P\sim43Q=\pm1$$

Let $\frac{A}{B}=\frac{43}{37}$ A=43 and B=37

Express $\frac{43}{37}$ into C. F. by finding H. C. F. and we get

$$\frac{A}{B}=1+\frac{1}{6+}\;\frac{1}{6}$$

Let $\frac{P}{Q}$ be the last but one convergent$=1+\frac{1}{6}=\frac{7}{6}$

$\therefore\;\; P=7, Q=6 \quad \therefore\;\; QA\sim PB=6.43\sim 7.37=\pm 1$

***Ex. 2. Find two polynomials P and Q such that**

If $A=3x^3+4x^2-3x+7$

and $B=x^2+2x-1$ tben $QA-PB=1$

Express $\frac{A}{B}$ as a continued fraction which is

$$3x-2+\frac{1}{\left(\frac{x}{4}+\frac{3}{16}\right)+}\quad\frac{1}{\left(-\frac{64}{31}x-\frac{80}{31}\right)}$$

The last but one convergent $\frac{P'}{Q'}=3x-2+\frac{1}{\frac{x}{4}+\frac{3}{16}}$

$$=\frac{\frac{3}{4}x^2+\frac{1}{16}x+\frac{5}{8}}{\frac{1}{4}x+\frac{3}{16}}$$

Let $P'=\frac{3}{4}x^2+\frac{1}{16}x+\frac{5}{8}$ and $Q'=\frac{1}{4}x+\frac{3}{16}$

$$\therefore\;\; AQ'-BP'=(3x^3+4x^2-3x+7)\left(\frac{1}{4}x+\frac{3}{16}\right)$$

$$-(x^2+2x-1)\left(\frac{3}{4}x^2+\frac{1}{16}x+\frac{5}{8}\right)=\frac{31}{16}$$

$$\therefore\;\; \frac{16}{31}AQ'-\frac{16}{31}BP'=1$$

But $QA-PB=1$ given $\;\therefore\;$ Comparing we get

$$Q=\frac{16}{31}Q'=\frac{16}{31}\left(\frac{x}{4}+\frac{3}{16}\right)=\frac{4x+3}{31}$$

$$P=\frac{16}{31}P'=\frac{16}{31}\left(\frac{3}{4}x^2+\frac{1}{16}x+\frac{5}{8}\right)=\frac{12x^2+x+10}{31}$$

§ 7. Recurring Continued Fraction.

If in an infinite continued fraction, a fixed number of quotients are, after some stage, repeated again and again, the fraction is known as a recurring continued fraction. The following are recurring C. F.

$$1+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\cdots\cdots\infty$$

$$a+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots\cdots\infty$$

A simple example of a recurring continued fraction is

$$a+\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots\cdots\infty\cdots\cdots(A)$$

in which the quotient a is repeated again and again.

If x be the value of the continued fraction A, then

$$x=a+\frac{1}{x}$$

or $\qquad x^2-ax-1=0$

$\therefore\qquad x=\frac{1}{2}\{a+\sqrt{(a^2+4)}\}$ always taking the +ive sign.

Ex. 1. If $\quad x=a+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots\cdots\infty$

and $y=b+\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots\cdots\infty$, then $bx=ay$

Now $\quad x=a+\frac{1}{y}\quad$ or $\quad xy=ay+1\ldots\ldots(A)$

Also $\quad y=b+\frac{1}{x}\quad$ or $\quad xy=bx+1\ldots\ldots(B)$

$\therefore\quad ay+1=bx+1\qquad$ or $\qquad ay=bx$ from (A) and (B)

***Ex. 2.** Evaluate $X=4+\frac{1}{7+}\,\underset{\times}{\frac{1}{1+}}\,\frac{1}{2+}\,\underset{\times}{\frac{1}{3+}}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots\cdots\infty$

where × × represents the beginning and the end of the cycle of partial quotients.

$$X=4+\frac{1}{7+y}$$

where $$y=\underset{\times}{\frac{1}{1+}}\;\frac{1}{2+}\;\underset{\times}{\frac{1}{3+}}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{3+}\cdots\cdots$$

or $$y=\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{3+y}$$

$$=\frac{1}{1+\cfrac{1}{2+\cfrac{1}{3+y}}}=\frac{1}{1+\cfrac{3+y}{7+2y}}=\frac{7+2y}{10+3y}$$

$$\therefore\quad 3y^2+8y-7=0 \quad\text{or}\quad y=\frac{-4+\sqrt{(37)}}{3}$$

We have rejected the negative sign as it will make the value of y −ive which is not possible.

$$\therefore\quad X=4+\frac{1}{7+\cfrac{\sqrt{(37)}-4}{3}}=4+\frac{3}{\sqrt{(37)}+17}$$

$$=\frac{71+4\sqrt{(37)}}{17+\sqrt{(37)}}$$

$$=\frac{71+4\sqrt{(37)}}{17+\sqrt{(37)}}\cdot\frac{17-\sqrt{(37)}}{17-\sqrt{(37)}}=\frac{1059-3\sqrt{(37)}}{252}$$

***Ex. 2. (b) Evaluate** $X=1+\frac{1}{2+}\;\frac{1}{3+}\;\frac{1}{2+}\;\frac{1}{3+}\cdots\cdots$

(Agra 1946)

Ans. $\frac{-1+\sqrt{(15)}}{2}$

§ 8, General Continued Fraction.

For the continued fraction,

$$a_1+\frac{b_2}{a_2+}\;\frac{b_3}{a_3+}\;\frac{b_4}{a_4+}\cdots\cdots$$

Art. 8 and Ex. 1 may be omitted.

The successive convergents are defined to be

$$a_1,\quad a_1+\frac{b_2}{a_2},\quad a_1+\frac{b_2}{a_2+}\,\frac{b_3}{a_3},\ \ldots\ldots\ldots$$

The law of formation of successive convergents is

$$p_n=a_np_{n-1}+b_n\,p_{n-2}$$
$$q_n=a_nq_{n-1}+b_n\,q_{n-2}$$

Putting $b=1$ we get the law of formation for simple continued fraction.

Sometimes it will be convenient to proceed from first principles.

Ex. 1. Find the *n*th convergent to

$$\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\cdots\cdots$$

$$\frac{p_1}{q_1}=\frac{1}{2},\quad \frac{p_2}{q_2}=\frac{1}{2-\frac{1}{2}}=\frac{2}{3}$$

$$\frac{p_3}{q_3}=\frac{1}{2-\dfrac{1}{2-\frac{1}{2}}}=\frac{1}{2-\frac{2}{3}}=\frac{3}{4}$$

From above it is clear that $\dfrac{p_n}{q_n}=\dfrac{n}{n+1}$

Again $\dfrac{p_{n+1}}{q_{n+1}}=\dfrac{1}{2-}\,\dfrac{1}{2-}\,\dfrac{1}{2-}\cdots\cdots(n+1)$ quotients

$$=\frac{1}{2-}\left(\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\cdots\cdots n\text{ quotients}\right)$$

$$=\frac{1}{2-}\,\frac{p_n}{q_n}=\frac{1}{2-\dfrac{p_n}{q_n}}$$

$$=\frac{1}{2-\dfrac{n}{n+1}}=\frac{n+1}{n+2}$$

Hence the above formula is general.

EXERCISE 2

1. Prove the following :—

(a) $\dfrac{2345}{1234}=\left[1+\dfrac{1}{1+}\,\dfrac{1}{9+}\,\dfrac{1}{30+}\,\dfrac{1}{1+}\,\dfrac{1}{3}\right]$

(b) $\cdot 3029=\left[\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{10}\right]$.

(c) $31\frac{76}{123}=\left[31+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3}\right]$

(d) $\frac{427}{2166}=\left[\frac{1}{5+}\,\frac{1}{13+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3}\right]$

2. Prove the following :—

(a) $\sqrt{(33)}=\left[5+\frac{1}{\underset{\times}{1+}}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{\underset{\times}{10+}}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{10+}\cdots\right]$

Proceed as in Ex. 2 page 48.

(b) $\sqrt{7}=\left[2+\frac{1}{\underset{\times}{1+}}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\underset{\times}{4+}}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\right.$
$\left.\frac{1}{4+}\cdots\cdots\right]$

*(c) $\sqrt{(10)}=\left[3+\frac{1}{\underset{\times}{6+}}\,\frac{1}{\underset{\times}{6+}}\,\frac{1}{6+}\cdots\cdots\right]$ (Agra 1949)

(d) $\sqrt{(12)}=\left[3+\frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{2+}\,\frac{1}{6+}\cdots\cdots\right]$

*(e) $\sqrt{2}=\left[1+\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots\cdots\right]$ (Agra 1944)

(f) $\sqrt{(14)}=\left[3+\frac{1}{1+}\,\frac{1}{\underset{\times}{2+}}\,\frac{1}{\underset{\times}{1+}}\,\frac{1}{6}\cdots\cdots\right]$ (Nagpur 1933)

3 (a) **Express $\sqrt{(a^2+1)}$ as a continued fraction** (I. C. S. 1941)

$\left[a+\frac{1}{2a+}\,\frac{1}{2a+}\,\frac{1}{2a+}\cdots\cdots\right]$ **Ans.**

(b) **Express as a continued fraction**

$$\frac{1+3\sqrt{5}}{2}$$

(i) 3 being the integer next less than $\frac{1+3\sqrt{5}}{2}$

$$\therefore \quad \frac{1+3\sqrt{5}}{2}=3+\frac{3\sqrt{5}-5}{2}$$

$$=3+\cfrac{1}{\cfrac{2}{3\sqrt{5}-5}}=3+\cfrac{1}{\cfrac{2(3\sqrt{5}+5)}{45-25}}$$

$$=3+\cfrac{1}{\cfrac{3\sqrt{5}+5}{10}}\ldots\ldots(A)$$

(ii) Again 1 being the integer next less than $\frac{3\sqrt{5}+5}{10}$

$$\therefore \quad \frac{3\sqrt{5}+5}{10}=1+\frac{3\sqrt{5}-5}{10}=1+\cfrac{1}{\cfrac{10}{3\sqrt{5}-5}}$$

$$=1+\cfrac{1}{\cfrac{10(3\sqrt{5}+5)}{45-25}}=1+\cfrac{1}{\cfrac{3\sqrt{5}+5}{2}}\ldots\ldots(B)$$

(iii) Again 5 being the integer next less than $\frac{3\sqrt{5}+5}{2}$

$$\therefore \quad \frac{3\sqrt{5}+5}{2}=5+\frac{3\sqrt{5}-5}{2}$$

$$=5+\cfrac{1}{\cfrac{2}{3\sqrt{5}-5}}=5+\cfrac{1}{\cfrac{2(3\sqrt{5}+5)}{45-25}}$$

$$=5+\cfrac{1}{\cfrac{3\sqrt{5}+5}{10}}$$

And now we arrive at A.

$$\therefore \quad \frac{1+3\sqrt{5}}{2} = 3 + \underset{\times}{\frac{1}{1+}}\ \underset{\times}{\frac{1}{5+}}\ \frac{1}{1+}\ \frac{1}{5+} \cdots\cdots$$

where × × represents the beginning and the end of the cycle of partial quotients.

(c) **Express as continued fraction** $\dfrac{2x^3-4x^2+2x-3}{x^2-2x+3}$

$$\left\{ 2x + \frac{1}{\left(-\frac{x}{4}+\frac{11}{16}\right)+}\ \frac{1}{\left(\frac{-64}{81}x-\frac{48}{81}\right)} \right\} \quad \text{(Ans.)}$$

*(d) **Find the 3rd convergent of** $\dfrac{n^4+n^2-1}{n^3+n^2+n+1}$

(Sagar 1948)

$$\left[(n-1) + \frac{1}{(n+1)+}\ \frac{1}{(n-1)+}\ \frac{1}{(n+1)} \right]$$

$$\frac{p_3}{q_3} = n^3 \quad \frac{n^2+n-1}{n^2}$$

(e) **Find the 3rd convergent of** $\dfrac{a^3+6a^2+13a+10}{a^4+6a^3+14\imath^2+15a+7}$

$$\left[\frac{1}{a+}\ \frac{1}{(a+1)+}\ \frac{1}{(a+2)+}\ \frac{1}{(a+3)} \right]$$

$$\frac{p_3}{q_3} = \frac{a^2+3a+3}{a^3+3a^2+4a+2}$$

4. (a) **Express** $\dfrac{x^2+x+1}{x^2+1}$ **as a simple continued fraction and find P and Q two polynomials such that**

$$Q(x^2+x+1) - P(x^2+1) = -1.$$

Proceed as in Ex 2 page 68

$\left[\text{C. F. is } 1 + \frac{1}{x+}\ \frac{1}{x} \right]$, $Q = x$ and $P = x+1$

***(b) Given that** $A=2x^3-4x^2+2x-3$

$$B=x^2-2x+3$$

Find two polynomials p and Q such that

$$AQ-PB=1$$

C. F. is found in Q. 3 (c) for P and Q proceed as in Ex. 2 Page 68.

$$Q=\frac{4x-11}{81} \text{ and } P=\frac{16+22x-8x^2}{81}$$

***(c) Express $\frac{763}{396}$ as a continued fraction and find the least values of P and Q such that**

$$396P-763Q=12$$ **(Agra 1940)**

Finding H. C. F. of 763 and 396 the successive quotients are found out to be

1, 1, 12, 1, 1, 1, 9.

$$\therefore \text{ C. F. is } 1+\frac{1}{1+}\;\frac{1}{12+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{9}$$

$$\frac{p_1}{q_1}=\frac{1}{1},\ \frac{p_2}{q_2}=1+\frac{1}{1}=\frac{2}{1},\ \frac{p_3}{q_3}=\frac{a_3p_2+p_1}{a_3q_2+q_1}=\frac{25}{13}.$$

$$\frac{p_4}{q_4}=\frac{27}{14},\ \frac{p_5}{q_5}=\frac{52}{27},\ \frac{p_6}{q_6}=\frac{79}{41}$$

The last but one convergent $=\frac{p_6}{q_6}=\frac{79}{41}=\frac{P'}{Q'}$

$$396\ P'-763\cdot Q'=396(79)-763(41)=1$$

$$\text{and } 396P-763Q=12$$

Conparing $\frac{P}{79}=\frac{Q}{41}=12$

$\therefore\ P=12\times 79=948,\quad Q=12\times 41=492$

***(d) Express $\frac{157}{68}$ as a simple continued fraction with an odd number of quotients and hence find the least value of x and y which satisfy the equation $68x-157y=1$ (Agra 1954)**

***5 (a) Find a series of fractions converging to ·24226, the excess in days of the true tropical year over 365 days.**

Find successive convergents to $\frac{24226}{100000}$ which by prop. 11 is the required series of fractions.

***(b) A kilometer is very nearly equal to ·62138 miles, show that the fraction**

$\frac{5}{8}$, $\frac{18}{29}$, $\frac{23}{37}$, $\frac{64}{103}$ are successive convergents approximating to the ratio of a kilometer to a mile and show that 23 miles are nearly equal to 37 kilometers.

Find successive convergents to $\frac{\text{kilometer}}{\text{mile}}=\frac{62138}{100000}$ one of which is $\frac{23}{37}$ and by prop. 11 this convergent is nearly equal to fraction itself.

$\therefore$ 37 kilometers $=$ 23 miles.

(a) A meter is 39·37079 inches show by the theory of continued fraction that 32 meters is nearly equal to 35 yds.

$$1 \text{ meter}=\frac{3937079}{100000} \text{ inches}=\frac{3937079}{3600000} \text{ yds.}$$

Proceeding as in (b) $\frac{p_4}{q_4}=\frac{35}{32}$. Hence 32 meters $=$ 35 yds.

***(d) Two scale pans of equal length are divided into 162 and 209 equal parts respectively. Show that if their zero points be coincident, the 31st division of first nearly coincides with 40th division of the other.**

(Nagpur 1950)

Since the scale pans are of equal length and their zero points coincident.

$\therefore$ 162 parts of 1st scale $=$ 209 parts of the other

Or One part of 1st scale $=\frac{209}{162}$ parts of the other

Now fourth convergent of $\frac{209}{162}$ is found out to be $\frac{40}{31}$ which by prop. 11 is approximately equal to fraction itself.

***(e) A lunar month consists of 29·53 days and a tropical year of 365·24 days. Show by forming**

successive convergents that there are nearly 99 lunar months in 8 tropical years or 235 lunar months in 19 tropical years. (Agra 1934, Rajputana 1953)

$$\frac{1.\text{Tropical year}}{1.\text{lunar month}}=\frac{365.2 \text{ days}}{29.53 \text{ days}}=\frac{36524}{2953}$$

$$=12+\frac{1}{2+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{2+}\cdots\cdots$$

The successive convergents are

$$12, \frac{25}{2}, \frac{37}{2}, \frac{99}{8}, \frac{235}{19} \text{ etc.}$$

6. Show by the theory of continued fractions.

(a) That $\frac{99}{70}$ differs from 1.41421 by a quantity less than $\frac{1}{11830}$

Convert 1·41421 into C. F. and we find that

$\frac{p_6}{q_6}=\frac{99}{70}$ and $\frac{p_7}{q_7}=\frac{239}{169}$ and by prop. 12 c. error is less than $\frac{1}{q_n q_{n+1}}$

(b) Find the limits to error in taking $\frac{222}{203}$ yds. as equivalent to a meter, given that a meter is equal to 1·0936 yds.

$1{\cdot}0936=\frac{10936}{10000}$. Finding H. C. F. the simple continued fraction is found out to be

$$1+\frac{1}{10+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{6}$$

and successive convergents are

$$1, \frac{11}{10}, \frac{12}{11}, \frac{35}{32}, \frac{222}{203}, \frac{1367}{1250}$$

Now if we take $\frac{222}{203}$ as equivalent to a meter than by prop. 12 B error is greater than

$$\frac{1}{q_n(q_{n+1}+q_n)} \text{ or } > \frac{1}{2(q_{n+1})^2}$$

and from 12. D and E error is

$$< \frac{1}{a_{n+1}q_n^2} \text{ or } < \frac{1}{q_n^2}$$

$$\therefore \text{ error is } > \frac{1}{2(1250)^2} < \frac{1}{(203)^2}$$

***(c) Find good approximation to the ratio of circumference of a circle to that of its diameter. Given that $\pi=3\ 1{,}416$.**

Proceed as in solved example Page 67 and $\frac{355}{113}$ is a good approximation, the error by 12 D is less than $\frac{1}{(113)^2\ .\ 292}$

Circumference of a circle $=2a\pi$.

(d) Find an approximation to

$$1+\frac{1}{3+}\ \frac{1}{5+}\ \frac{1}{7+}\ \frac{1}{9+}\ \frac{1}{19+}\cdots\cdots$$

Which differs from the true value by less than ·0001.

Chose $\frac{p_4}{q_4}=\frac{151}{115}$ as an approximation which by prop. 12 E gives error $< \frac{1}{q_n^2}$ i. e. $< \frac{1}{(115)^2} < \frac{1}{(100)^2}$ or $<$ ·0001

(e) Express $\sqrt{(11)}$ into a simple continued fraction and show that the error in taking fourth convergent equivalent to $\sqrt{(11)}$ is less than ·00005.

$$\sqrt{(11)}=3+\frac{1}{3+}\ \frac{1}{6+}\ \frac{1}{3+}\ \frac{1}{6+}\cdots\cdots$$

× ×

and $\frac{p_4}{q_4}=\frac{199}{60}$

$\therefore$ By prop. 12·D error is less than

$$\frac{1}{a_{n+1}q_n^2} \text{ i. e. } < \frac{1}{6(60)^2} \text{ or } < \frac{1}{21600}$$
$$< \frac{1}{20000} < \cdot 00005$$

***7. If $\frac{p_n}{q_n}$ is the nth convergent of the continued fraction $\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\ldots\ldots$ show that**

(a) $p_n^2+p_{n+1}^2=p_{n-1}p_{n+1}+p_np_{n+2}$ **(Agra 1950)**

(b) $p_n=q_{n-1}$ **(Raj. 1951, Agra 50, 43, Sagar 37, 50)**

(c) $q_n=ap_n+p_{n-1}$ **(Luck. 43 Supp.)**

From (a) we get $p_{n+1}(p_{n+1}-p_{n-1})=p_n(p_{n+2}-p_n)$

Now all the quotients in the given continued fraction are equal to a i.e. $a_{n+1}=a_{n+2}=a$.

or $p_{n+1}(a_{n+1}p_n)=p_n(a_{n+2}p_{n+1})$

$\therefore$ $p_{n+1}(a\,p_n)=p_n(a\,p_{n+1})$ which is true.

(b) $p_n=q_{n-1}\ldots\ldots\ldots$(B)

$$\frac{p_1}{q_1}=\frac{1}{a},\quad \frac{p_2}{q_2}=\frac{1}{a+\frac{1}{a}}=\frac{a}{a^2+1},\quad \frac{p_3}{q_3}=\frac{a^2+1}{a^3+2a}$$

From above we find tbat $p_3=a^2+1=q_2$

$p_2=a=q_1$

i.e. when $n=2$, 3, 4 and so on then (B) holds. Let us assume that it holds. For $n=n$, i.e. $p_n=q_{n-1}$

Now $p_{n+1}=a\,p_n+p_{n-1}=aq_{n-1}+q_{n-2}$ (by assumption)

$=q_n$

$\because$ a can be regarded as a_n

Hence we find that (B) holds for $n=n+1$.

$\therefore$ It is universally true.

(c) $q_n=ap_n+p_{n-1}$

By (B) $p_n=q_{n-1}$ and $p_{n-1}=q_{n-2}$

$\therefore$ $ap_n+p_{n-1}=aq_{n-1}+q_{n-2}=q_n$

$\because$ a can be regarded as a_n.

Questions from 7 to 17 are all very important and should be done thoroughly.

Hence proved.

***8. If $\frac{p_n}{q_n}$ is the nth convergent to the continued fraction $\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\dots\dots$ show that**

(a) (1) $p_{2n+2}=bq_{2n}+p_{2n}$ and $q_{2n+2}=(ab+1)q_{2n}+ap_{2n}$

(Luck. 40. Agra 48 Supp.)

(2) $q_{2n}=p_{2n+1}$ and $bq_{2n-1}=ap_{2n}$

(Allahabad 38, 40, 50, Agra 45. Raj. 49, 52)

(1) We observe from the given continued fraction that all odd quotients are equal to a and all even quotients are equal to b.

$$\frac{p_{2n}}{q_{2n}}=\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\dots\dots \text{ 2n quotients}$$

$$\frac{p_{2n+2}}{q_{2n+2}}=\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\dots\dots(2n+2)\text{ quotients}$$

$$=\frac{1}{a+}\,\frac{1}{b+}\left(\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\dots\dots\dots \text{ 2n quotients}\right)$$

$$=\frac{1}{a+}\,\frac{1}{b+}\left(\frac{p_{2n}}{q_{2n}}\right)$$

$$=\frac{1}{a+\cfrac{1}{b+\cfrac{p_{2n}}{q_{2n}}}}=\frac{1}{a+\cfrac{q_{2n}}{bq_{2n}+p_{2n}}}$$

$$=\frac{bq_{2n}+p_{2n}}{a(bq_{2n}+p_{2n})+q_{2n}}=\frac{bq_{2n}+p_{2n}}{(ab+1)q_{2n}+ap_{2n}}\,.$$

By property no. 2 each convergent is in its lowest term

∴ $p_{2n+2}=bq_{2n}+p_{2n}$ and $q_{2n+2}=(ab+1)q_{2n}+ap_{2n}$

which proves (1)

(2) Now $p_{2n+2}-p_{2n}=bq_{2n}$ from (1)

or $a_{2n+2}p_{2n+1}+p_{2n}-p_{2n}=bq_{2n}$

or $bp_{2n+1}=bq_{2n}$ ∵ a_{2n+2}=even quotient$=b$

∴ $p_{2n+1}=q_{2n}$......(B), which proves (2)

Again expanding relation B and remembering that odd quotients are a and even are b we get

$$ap_{2n}+p_{2n-1}=bq_{2n-1}+q_{2n-2}$$

Putting $n=n-1$ in B we get $p_{2n-1}=q_{2n-2}$

$\therefore \quad ap_{2n}=bq_{2n-1}$ Hence proved.

***(b) Prove that $p_{n+2}-(ab+2)p_n+p_{n-2}=0$**

(Rangoon 1937, Andhra 1938, Agra 1941, 51, Lucknow 1952, Benaras 1948).

and $q_{n+2}-(ab+2)q_n+q_{n-2}=0$

Now $p_{n+2}=a_{n+2}p_{n+1}+p_n$

$$=a_{n+2}(a_{n+1}p_n+p_{n-1})+p_n$$

$$=(a_{n+2}a_{n+1}+1)p_n+a_{n+2}p_{n-1}$$

From the given continued fraction we can say that $a_{n+2}=a_n$.

[1st, 3rd, 5th, and so on partial quotients are equal and 2nd, 4th, 6th, and so on partial quotients are equal]

$$=(a_{n+2}a_{n+1}+1)p_n+a_np_{n-1}$$

$$=(a_{n+2}a_{n+1}+1)p_n+p_n-p_{n-2}$$

[$\because \; p_n=a_np_{n-1}+p_{n-2}$]

$$=(a_{n+2}a_{n+1}+2)p_n-p_{n-2}$$

Now $a_{n+2}\cdot a_{n+1}$ being the product of two consecutive quotients must be equal to ab because the convergents are alternately equal to a and b.

$\therefore \quad p_{n+2}=(ab+2)p_n-p_{n-2}$

Or $p_{n+2}-(ab+2)p_n+p_{n-2}=0$

Similarly we can prove the second part :—

***8 (c) If $\frac{p_n}{q_n}$ is the nth convergent of the continued fraction.**

$\frac{1}{a+}\;\frac{1}{b+}\;\frac{1}{a+}\;\frac{1}{b+}$**.........prove that** **(Agra 1954)**

$p_n-(ab+2)p_{n-2}+p_{n-4}=0$

$q_n-(ab+2)q_{n-2}+q_{n-4}=0$, Proceed as above. Start with p_n.

✓*9. If $\frac{p_n}{q_n}$ be the nth convergent of

$$\underset{\times}{\frac{1}{a+}}\ \frac{1}{b+}\ \underset{\times}{\frac{1}{c+}}\ \frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+}\cdots\cdots\cdots$$

Show that $p_{3n+3}=bp_{3n}+(bc+1)q_{3n}$

$$\frac{p_{3n+3}}{q_{3n+3}}=\underset{\times}{\frac{1}{a+}}\ \frac{1}{b+}\ \underset{\times}{\frac{1}{c+}}\cdots\cdots(3n+3)\text{ quotients}$$

$$\frac{p_{3n}}{q_{3n}}=\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+}\cdots\cdots\cdots 3n\text{ quotients.}$$

$$\therefore\ \frac{p_{3n+3}}{q_{3n+3}}=\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+}\left(\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+}\cdots\cdots\cdots 3n\text{ quotients}\right)$$

$$=\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+}\ \left(\frac{p_{3n}}{q_{3n}}\right)$$

$$=\frac{1}{a+}\ \frac{1}{b+\dfrac{1}{c+\dfrac{p_{3n}}{q_{3n}}}}=\frac{1}{a+}\ \frac{1}{b+\dfrac{q_{3n}}{cq_{3n}+p_{3n}}}$$

$$=\frac{1}{a+\dfrac{cq_{3n}+p_{3n}}{(bc+1)q_{3n}+bp_{3n}}}=\frac{(bc+1)q_{3n}+bp_{3n}}{(abc+a+c)q_{3n}+(ab+1)p_{3n}}$$

Again by prop. 2 the convergents are in their lowest terms.

$\therefore\ p_{3n+3}=(bc+1)q_{3n}+bp_{3n}$

***10. If $\frac{\mathrm{M}}{\mathrm{N}}$, $\frac{\mathrm{P}}{\mathrm{Q}}$, $\frac{\mathrm{R}}{\mathrm{S}}$ are the nth, (n – 1)th, and (n – 2)th convergents to the continued fractions.**

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\cdots,\quad \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\cdots$$

and $\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots\cdots$ respectively

show that $M=a_2P+R \qquad N=(a_1a_2+1)P+a_1R$

(Alld. 36)

$\frac{M}{N}$ being the nth convergent of the first C. F.

$$=\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\cdots\frac{1}{a_n}$$

$\frac{P}{Q}$ being the $(n-1)$th convergent of the second C. F.

$$=\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\cdots\frac{1}{a_n}$$

$\frac{R}{S}$ being the $(n-2)$th convergent of the third C. F.

$$=\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots\cdots\frac{1}{a_n}$$

$$\frac{M}{N}=\frac{1}{a_1+}\,\frac{P}{Q}=\frac{1}{a_1+\frac{P}{Q}}=\frac{Q}{a_1Q+P}$$

$$\frac{P}{Q}=\frac{1}{a_2+}\,\frac{R}{S}=\frac{1}{a_2+\frac{R}{S}}=\frac{S}{a_2S+R}$$

By prop. no. 2 the convergents are in their lowest terms.

$$\therefore\quad M=Q \qquad N=a_1Q+P$$

$$P=S \qquad Q=a_2S+R$$

or $M=Q=a_2S+R=a_2P+R$

$$N=a_1Q+P=a_1(a_2S+R)+P=a_1a_2P+a_1R+P$$

$$=(a_1a_2+1)P+a_1R.$$ Hence proved.

11. Prove

$$a\left(x_1+\frac{1}{ax_2+}\,\frac{1}{x_3+}\,\frac{1}{ax_4}\ldots\ldots 2n\text{ quotients}\right)$$

$$=ax_1+\frac{1}{x_2+}\,\frac{1}{ax_3+}\,\frac{1}{x_4+}\ldots\ldots 2n\text{ quotients}$$

$$\text{L.H.S.}=ax_1+\cfrac{a}{ax_2+\cfrac{1}{x_3+\cfrac{1}{ax_4\ldots\ldots}}}$$

$$=ax_1+\cfrac{1}{\frac{1}{a}\left\{ax_2+\cfrac{1}{x_3+\cfrac{1}{ax_4}}\right\}\ldots\ldots}$$

$$=ax_1+\cfrac{1}{x_2+\cfrac{1}{a\left\{x_3+\cfrac{1}{ax_4}\right\}\ldots\ldots}}$$

$$=ax_1+\cfrac{1}{x_2+\cfrac{1}{ax_3+\cfrac{1}{x_4+\ldots\ldots}}}=\text{R.H.S.}$$

***12. If** $\frac{p_n}{q_n}$ **is the nth convergent to the continued fraction** $a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\ldots\ldots$ **then prove**

***(a)** $(p_n^2-q_n^2)(p_{n-1}^2-q_{n-1}^2)=(p_np_{n-1}-q_nq_{n-1})^2-1$

(Agra 33, Andhra 41)

On multiplication the L. H. S. becomes

$$(p_n^2p_{n-1}^2+q_n^2q_{n-1}^2)-(p_n^2q_{n-1}^2+q_n^2p_{n-1}^2)$$

Subtracting $2p_np_{n-1}q_nq_{n-1}$ from both the brackets, we get

L. H. S. $(p_np_{n-1}-q_nq_{n-1})^2-(p_nq_{n-1}-q_np_{n-1})^2$

$$=(p_np_{n-1}-q_nq_{n-1})^2-[(-1)^n]^2$$
$$=(p_np_{n-1}-q_nq_{n-1})^2-(-1)^{2n}$$
$$=(p_np_{n-1}-q_nq_{n-1})^2-1 \qquad \because \ (-1)^{2n}=1$$

*(b) $(p_nq_n+p_{n-1}q_{n-1})-(p_{n-2}q_n+p_{n-1}q_{n+1})$
$$=(a_n-a_{n+1})p_{n-1}q_n$$

L. H. S. is $q_n(p_n-p_{n-2})-p_{n-1}(q_{n+1}-q_{n-1})$

$$=q_na_np_{n-1}-p_{n-1}a_{n+1}q_n$$
$$=(a_n-a_{n+1})p_{n-1}q_n$$

Hence proved.

*(c) $$\frac{p^2_n+q^2_n}{p^2_{n-2}+q^2_{n-2}}=\frac{(p_np_{n-1}+q_nq_{n-1})^2+1}{(p_{n-1}p_{n-2}+q_{n-1}q_{n-2})^2+1}$$

Now

$1=(-1)^{2n}=[(-1)^n]^2=(p_nq_{n-1}-q_np_{n-1})^2$ prop. no. 1

$1=(-1)^{2n-2}=[(-1)^{n-1}]^2=(p_{n-1}q_{n-2}-q_{n-1}p_{n-2})^2$

$$\therefore \text{ R.H.S.}=\frac{(p_np_{n-1}+q_nq_{n-1})^2+(p_nq_{n-1}-q_np_{n-1})^2}{(p_{n-1}p_{n-2}+q_{n-1}q_{n-2})^2+(p_{n-1}q_{n-2}-q_{n-1}p_{n-2})^2}$$
$$=\frac{p^2_n(p^2_{n-1}+q^2_{n-1})+q^2_n(p^2_{n-1}+q^2_{n-1})}{p^2_{n-1}(p^2_{n-2}+q^2_{n-2})+q^2_{n-1}(p^2_{n-2}+q^2_{n-2})}$$
$$=\frac{(p^2_n+q_n^2)(p^2_{n-1}+q^2_{n-1})}{(p^2_{n-2}+q^2_{n-2})(p^2_{n-1}+q^2_{n-1})}=\frac{p^2_n+q^2_n}{p^2_{n-2}+q^2_{n-2}}$$

***13. Show that the difference between the first and nth convergent is numerically equal to**

$$\frac{1}{q_1q_2}-\frac{1}{q_2q_3}+\dots+\frac{(-1)^n}{q_nq_{n-1}}$$

(Calcutta 1931, Madras 1941, Travancore 1948, Allahabad 1950),

Now $$\frac{p_2}{q_2}-\frac{p_1}{q_1}=\frac{p_2q_1-q_2p_1}{q_2q_1}=\frac{(-1)^2}{q_1q_2}=\frac{1}{q_1q_2}$$

$$\frac{p_3}{q_3}-\frac{p_2}{q_2}=\frac{p_3q_2-q_3p_2}{q_3q_2}=\frac{(-1)^3}{q_3q_2}=\frac{-1}{q_3q_2}$$

Similarly

$$\frac{p_4}{q_4}-\frac{p_3}{q_3}=\frac{p_4q_3-q_4p_3}{q_4q_3}=\frac{(-1)^4}{q_4q_3}=\frac{1}{q_4q_3}$$

……

$$\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{(-1)^n}{q_nq_{n-1}}$$

Adding all these and cancelling diagonally we get

$$\frac{p_n}{q_n}-\frac{p_1}{q_1}=\frac{1}{q_1q_2}-\frac{1}{q_2q_3}+\frac{1}{q_3q_4}-\cdots\cdots,\frac{+(-1)^n}{q_nq_{n-1}}$$

✓*14. If $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$ be the last and last but one convergents of (Nagpur 49)

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\cdots\frac{1}{k+}\,\frac{1}{l}$$

[a, b, c......k, l are n numbers]

Show that

$$X=\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\cdots\frac{1}{k+}\,\frac{1}{l+}\;\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\cdots\frac{1}{k+}\,\frac{1}{l}$$

$$=\frac{p_nq_n+p_np_{n-1}}{q^2_n+p_nq_{n-1}}$$

Now $\frac{p_{n+1}}{q_{n+1}}=\frac{a_{n+1}p_n+p_{n-1}}{a_{n+1}q_n+q_{n-1}}$......(1)

where a_{n+1} is $(n+1)$th partial quotient which is a in the continued fraction X as a, b, c, ...k, l are n numbers.

Now if in $(n+1)$th convergent we in place of $(n+1)$th partial quotient put $(n+1)$th complete quotient we shall get the continued fraction itself. Now $(n+1)$th complete quotient is $(n+1)$th partial quotient+all that follows it.

$$=a+\frac{1}{b+}\,\frac{1}{c+}\cdots\cdots\frac{1}{k+}\,\frac{1}{l}\cdots\cdots(2)$$

But $\frac{p_n}{q_n}=\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\cdots\frac{1}{k+}\,\frac{1}{l}$ [n quotients]

$$\text{or }\ \frac{p_n}{q_n}=\frac{1}{a+\frac{1}{b+}\,\frac{1}{c+}\cdots\cdots\frac{1}{k+}\,\frac{1}{l}}$$

$$=\frac{1}{(n+1)\text{th complete quotient}}\ \text{from (2)}$$

Taking reciprocal we get

$$\frac{q_n}{p_n}=(n+1)\text{th complete quotient.}$$

Substitute in (i) $(n+1)$th complete quotient in place of a_{n+1} and thus $\frac{p_{n+1}}{q_{n+1}}$ becomes X i. e. the continued fraction.

$$\therefore\ X=\frac{\frac{q_n}{p_n}p_n+p_{n-1}}{\frac{q_n}{p_n}q_n+q_{n-1}}=\frac{q_np_n+p_np_{n-1}}{q^2n+p_nq_{n-1}}$$

***15.** (a) In the C. F. $a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\cdots\frac{1}{a_n}$, Prove

(1) $\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\cdots\frac{1}{a_1}$ **(Agra 32, 47, 53)**

(2) $\frac{q_n}{q_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\cdots\frac{1}{a_2}$ **(Luck. 43, Annamalai 50)**

(3) $$\left(a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\cdots\cdots\frac{1}{a_1}\right)-\left(a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\cdots\frac{1}{a_2}\right)=\frac{(-1)^n}{p_{n-1}q_{n-1}}$$

(Agra 1951)

See Ex. 2 P. 54.

(b) In the C. F. $\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\cdots\frac{1}{a_n}$, prove

$$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\cdots\frac{1}{a_2}$$

$$\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}} + \dots\dots \frac{1}{a_1}$$

Proceed exactly as in Ex. 2, page 54.

Note—The forms of C. F. in a and b should be carefully looked and their results should be committed to memory.

***16.** (a) $\frac{P}{Q} = a + \frac{1}{b+}\,\frac{1}{c+} \dots\dots \frac{1}{k+}\,\frac{1}{l} = \frac{p_n}{q_n}$ say. **Show**

$$\frac{1}{l+}\,\frac{1}{k+} \dots\dots \frac{1}{b+}\,\frac{1}{a} \sim \frac{1}{l+}\,\frac{1}{k+}$$

$$\dots\dots \frac{1}{c+}\,\frac{1}{b} = \frac{1}{PQ}$$

By Question 15 (a),

Let $A = \frac{1}{l+}\,\frac{1}{k+} \dots\dots \frac{1}{b+}\,\frac{1}{a}$

$$= \cfrac{1}{l + \frac{1}{k+}\,\frac{1}{b+}\,\frac{1}{a}} = \cfrac{1}{\frac{p_n}{p_{n-1}}} = \frac{p_{n-1}}{p_n}$$

Let $B = \frac{1}{l+}\,\frac{1}{k+} \dots\dots \frac{1}{c+}\,\frac{1}{b}$

$$= \cfrac{1}{l + \frac{1}{k+} \dots\dots \frac{1}{c+}\,\frac{1}{b}} = \cfrac{1}{\frac{q_n}{q_{n-1}}} = \frac{q_{n-1}}{q_n}$$

Now subtracting A and B, we get

$$A - B = \frac{p_{n-1}}{p_n} - \frac{q_{n-1}}{q_n}$$

$$= \frac{q_n p_{n-1} - p_n q_{n-1}}{p_n q_n} = \frac{-(-1)^n}{p_n q_n} = \frac{(-1)^{n+1}}{p_n q_n}$$

$$\therefore \quad A \sim B = \frac{1}{p_n q_n} = \frac{1}{PQ} \quad \because \quad P = p_n \quad Q = q_n$$

***17. (a)** If $\frac{p_n}{q_n}$ be the nth convergent, (Agra 1936)

Show that

$$X=\frac{1}{a_1+}\ \frac{1}{a_2+}\cdots\cdots+\frac{1}{a_{n-1}+}\ \frac{1}{a_n+}\ \frac{1}{a_n+}\ \frac{1}{a_{n-1}+}\cdots\cdots\frac{1}{a_2+}\ \frac{1}{a_1}$$

$$=\frac{p_{n-1}q_{n-1}+p_nq_n}{q^2_{n-1}+q^2_n}$$

Now $\frac{p_n}{q_n}=\frac{1}{a_1+}\ \frac{1}{a_2+}\cdots\cdots\frac{1}{a_{n-1}+}\ \frac{1}{a_n}$

and $\frac{p_{n+1}}{q_{n+1}}=\frac{a_np_n+p_{n-1}}{a_nq_n+q_{n-1}}$ where a_n is the $(n+1)$th as well as nth partial quotient of X.

Also $(n+1)$th complete quotient is

$$a_n+\frac{1}{a_{n-1}+}\ \frac{1}{a_{n-2}+}\ \frac{1}{a_{n-3}+}\cdots\cdots\frac{1}{a_2+}\ \frac{1}{a_1}$$

which by Q. 15 $(b)=\frac{q_n}{q_{n-1}}$

Now if in $(n+1)$th convergent above instead of a_n the $(n+1)$th partial quotient we put $\frac{q_n}{q_{n-1}}$ the $(n+1)$th complete quotient, then $(n+1)$th convergent will be the continued fraction X.

$$X=\frac{\frac{q_n}{q_{n-1}}\cdot p_n+p_{n-1}}{\frac{q_n}{q_{n-1}}\cdot q_n+q_{n-1}}=\frac{p_nq_n+p_{n-1}q_{n-1}}{q^2_n+q^2_{n-1}}$$

***17. (b)** If $\frac{P}{Q}=a+\frac{1}{a_1+}\ \frac{1}{a_2+}\cdots\cdots\frac{1}{a_n+}\ \frac{1}{x}$ prove that $\frac{d}{dx}\left(\frac{P}{Q}\right)=\pm\frac{1}{Q^2}$

The given continued fraction has $(n+2)$ quotients, the last being x and all others are constants.

Now $\frac{P}{Q}=\frac{p_{n+2}}{q_{n+2}}=\frac{a_{n+2}p_{n+1}+p_n}{a_{n+2}q_{n+1}+q_n}=\frac{xp_{n+1}+p_n}{xq_{n+1}+q_n}\ldots(1)$

$$\therefore\quad \frac{d}{dx}\left(\frac{P}{Q}\right)=\frac{(xq_{n+1}+q_n)p_{n+1}-(xp_{n+1}+p_n)q_{n+1}}{(xq_{n+1}+q_n)^2}$$

$$=\frac{p_{n+1}q_n-q_{n+1}p_n}{(xq_{n+1}+q_n)^2}=\frac{(-1)^{n+1}}{Q^2}=\pm\frac{1}{Q^2}$$

Since convergents are in their lowest terms we get from (1), $xq_{n+1}+q_n=Q$.

18. **If** $\frac{p_1}{q_1}, \frac{p_2}{q_2}$**......be the convergents to**

$$1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\ldots\ldots\frac{1}{n+}\ldots\ldots\infty$$

show that

$$p_n=(n-1)p_{n-1}+(n-1)p_{n-2}+(n-2)p_{n-3}+\ldots\ldots 3p_2+2p_1+2$$

Here successive partial quotients are 1, 2, 3 and so on

and $\frac{p_1}{q_1}=1$ and $\frac{p_2}{q_2}=1+\frac{1}{2}=\frac{3}{2}$

$\therefore\quad p_2=3$ and $p_1=1$

$p_n=np_{n-1}+p_{n-2}\quad \because\quad a_n=n$

$p_{n-1}=(n-1)p_{n-2}+p_{n-3}$

$p_{n-2}=(n-2)p_{n-3}+p_{n-4}$

$\underline{p_{n-3}=(n-3)p_{n-4}+p_{n-5}}$

....

....

$p_3=3p_2+p_1$

$p_2=2p_1+1\quad \because\quad p_2=3$ and $p_1=1$

Adding vertically we get

$p_n+p_{n-1}+p_{n-2}+p_{n-3}\ldots\ldots+p_3+p_2$

$$=[np_{n-1}+(n-1)p_{n-2}+(n-2)p_{n-3}+(n-3)p_{n-4}+\ldots\ldots 3p_2+2p_1]$$

$$+(p_{n-2}+p_{n-3}+p_{n-4}+\ldots\ldots p_2+p_1+1)$$

Cancel underlined from both sides,

or $$p_n+p_{n-1}=\{np_{n-1}+(n-1)p_{n-2}+(n-2)p_{n-3}+(n-3)p_{n-4}+\ldots\ldots 3p_2+2p_1+p_1+1\}$$

Taking p_{n-1} from L. H. S. to R. H. S.

$$p_n=(n+1)p_{n-1}+(n-1)p_{n-2}+(n-2)p_{n-3}+(n-3)p_{n-4}+\ldots\ldots 3p_2+2p_1+2$$

$\because$ $(p_1+1=1+1=2)$ Hence proved.

19. (a) If $x=\frac{a}{b+}\,\frac{c}{d+}\,\frac{a}{b+}\,\frac{c}{d+}\ldots\ldots$

and $y=\frac{c}{d+}\,\frac{a}{b+}\,\frac{c}{d+}\,\frac{a}{b+}\ldots\ldots$

show that $bx-dy=a-c.$

$$x=\frac{a}{b+y} \qquad bx+xy=a$$

$$y=\frac{c}{d+x} \qquad dy+xy=c$$

Subtracting $bx-dy=a-c.$

Hence proved.

(b) Show that if

$$x=y+\frac{1}{2y+}\,\frac{1}{2y+}\ldots\infty$$

then $$y=x-\frac{1}{2x-}\,\frac{1}{2x-}\ldots\ldots\infty$$

Let $\frac{1}{2y+}\,\frac{1}{2y+}\ldots\ldots\infty=A$

$$A=\frac{1}{2y+A}$$

or $A^2+2Ay=1$

$(A+y)^2=1+y^2$ $\therefore$ $A+y=\sqrt{(1+y^2)}$

or $A=\sqrt{(1+y^2)}-y$

$$x=y+A=y+\sqrt{(1+y^2)}-y=\sqrt{(1+y^2)}$$

or $x^2=1+y^2$ ∴ $\sqrt{(x^2-1)}=y$......(1)

Put $-\frac{1}{2x-}\,\frac{1}{2x-}\,\frac{1}{2x-}\cdots\cdots=B.$ ∴ $y=x+B$

$$B=-\frac{1}{2x+B} \text{ or } B^2+2xB=-1$$

$$(B+x)^2=-1+x^2$$

$$B+x=\sqrt{(x^2-1)}$$

$$B=\sqrt{(x^2-1)}-x$$

$$y=x+B=x+\sqrt{(x^2-1)}-x=\sqrt{(x^2-1)}$$

which is true by (1)

***(c) Show that**

$$7+\frac{1}{14+}\,\frac{1}{14+}\cdots\cdots\infty=5\left(1+\frac{1}{2+}\,\frac{1}{2+}\cdots\cdots\infty\right)$$

Proceeding as in Ex. 2 of page 69, show that L. H. S. $=5\sqrt{2}=$ R. H. S.

(d) Find the value of the continued fraction

$$X=1+\frac{1}{\underset{\times}{2+}}\,\frac{1}{\underset{\times}{3+}}\,\frac{1}{2+}\,\frac{1}{3+}\cdots\cdots\cdots$$

Proceed as in Ex. 2 of page 69 $\left[X=\frac{\sqrt{(15)}-1}{2}\right]$ Ans.

(e) Find the value of the continued fraction

$$\frac{1}{\underset{\times}{1+}}\,\frac{1}{1+}\,\frac{1}{\underset{\times}{4+}}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\cdots\cdots$$

$\left[-2+\sqrt{\frac{13}{2}}\right]$ Ans.

***(f) Find the value of**

$$\frac{1}{\underset{\times}{3+}}\,\frac{1}{2+}\,\frac{1}{\underset{\times}{1+}}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots\cdots$$

$\left[\frac{-9+\sqrt{(129)}}{8}\right]$ Ans.

(Raj. 1950)

CHAPTER III

INEQUALITY

Definition :—

§1. **A statement that two functions are unequal is called an inequality. (Read)**

Such a statement is more definite if one of the functions is said to be greater than or less than the other. For the sake of brevity we shall use the following symbols, to represent the words written against them.

$>$ represents greater than

$<$ represents less than

a is said to be greater than b if a−b is +ive and a is said to be less than b if a−b is −ive.

i. e. $5 > 2$ $\because$ $5-2=3$, *a* +ive quantity

$-3 < -1$ $\because$ $-3-(-1)=-2$, a −ive quantity.

§2. **Fundamental properties of inequality. (Read)**

(a) If $a > b$ and c is any +ive number then

(*i*) $a+c > b+c$ (ii) $a-c > b-c$

(iii) $ac > bc$ (iv) $\frac{a}{c} > \frac{b}{c}$

i. e. in an inequality we can always add, subtract, multiply and divide each side by the same +ive quantity, without effecting the sign of the inequality.

(b) If $a-c > b$ given

then by (a) we can add c to both sides.

$\therefore$ $a-c+c > b+c$ i. e. $a > b+c$

which means that a term on one side of an inequality can be transposed on the other side provided its sign is changed.

(c) If $a > b$ then $a-b$ is +ive (definition).

Hence $-(a-b)$ will be $-$ive.

$\therefore$ $(-a)-(-b)$ will be $-$ive.

$\therefore$ $-a < -b$ (by definition).

Which means that if the sign of all the terms of the inequality be changed, the sign of the inequality is also changed.

(d) If $a > b$ then $(a-b)$ is +ive (definition).

$\therefore$ $c(a-b)$ is also +ive.

but $-c(a-b)$ is $-$ive.

$\therefore$ $(-ac)-(-bc)$ is $-$ive.

$\therefore$ $-ac < -bc$ (by definition),

which means that if we multiply both sides of an inequality by some $-$ive quantity, the sign of the inequality must also be changed.

(e) If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$ and so on,

then $a_1+a_2+a_3+\ldots\ldots a_n > b_1+b_2+b_3+\ldots\ldots b_n$

and $a_1 \cdot a_2 \cdot a_3 \ldots\ldots a_n > b_1 \cdot b_2 \cdot b_3 \ldots\ldots b_n$

(f) If $a > b$ then $a^n > b^n$, provided n is +ive.

If $a > b$ then $\frac{1}{a} < \frac{1}{b}$ and $\frac{1}{a^n} < \frac{1}{b^n}$

(g) $a > b$

then $e^a > e^b$ and $\log a > \log b$

§3. Arithmetic mean of any two +ive quantities is greater than their Geometric mean. (Read)

We know that the square of every real number is +ive.

$\therefore$ $(a-b)^2$ is +ive i. e. > 0

or $a^2+b^2-2ab > 0$ or $a^2+b^2 > 2ab$

or $\frac{a^2+b^2}{2} > ab$. Now put $x=a^2$ and $y=b^2$

$\therefore$ $\frac{x+y}{2} > \sqrt{(xy)}$ Hence proved.

§ 4. If the sum of two +ive quantities is constant then their product is maximum when the quantities are equal, and if the product is constant, then their sum is minimum when quantities are equal. (Result)

Let us take a and b, two +ive quantities whose product ab is denoted by P and their sum $a+b$ by S.

Now $4ab=(a+b)^2-(a-b)^2$.

i.e. $4P=S^2-(a-b)^2$......(i)

(a) Now if **S is constant,** then from (i) P will be maximum when $(a-b)^2=0$ i.e. when $a=b$ which means when the quantities become equal.

(b) Again if **P is constant** then from (i) S will be minimum when $(a-b)^2=0$ i.e. when $a=b$ which means when the quantities become equal.

Illustration,

(i) **Consider a sum 20.** You can make a number of factors of the form

(16, 4) (12, 8) (15, 5) (10, 10)

Product of these factors whose sum in each case is equal to 20 i.e. constant is 64, 96, 75 and 100 respectively. We note that max. product is those of equal factors (10, 10).

Hence if the sum is constant then product of equal factors is greatest.

Note—(16, 4) has sum 20 but they are unequal.

Equal factors will be $\left(\frac{16+4}{2}, \frac{16+4}{2}\right)$ i.e. (10, 10) and sum is again 20.

(ii) Consider a product 100, you can make a number of factors of the form

(20, 5), (25, 4) (10, 10) (50, 2)

Sum of these factors whose product in each case is equal to 100, i.e. constant (given) is 25, 29, 20 and 52 respectively; we observe that minimum sum is those of equal factors (10, 10).

Hence if the product is constant then the sum of equal quantities is least.

***5. Find the greatest value of a product the sum of whose factors is constant and hence prove that arithmetic mean of n' +ive quantities is greater than their geometric mean. (Agra 28, 35, 37, Punjab 44, 51)**

Proof—Let there be n factors a, b, c.........K such that $a+b+c+d+......K=S$ (a constant)

Consider the product $abcd......K$. In this if we replace the two unequal quantities a and b by two equal quantities $\frac{a+b}{2}$, $\frac{a+b}{2}$ then the value of the product will be increased (by art. 4 a) without affecting the sum of the factors, because the factors a, b, c, d,......K have also a sum

$$=a+b+c+d......K=S$$

and the factors $\frac{a+b}{2}$, $\frac{a+b}{2}$, c, d...K has a sum

$$=a+b+c+d+......K=S$$

Similarly we can further increase the value of the product by replacing two unequal factors c and d by $\frac{c+d}{2}$, $\frac{c+d}{2}$ without causing any change in the sum of the factors which will be constant.

Hence so long as any two of the factors a, b, c, d...

...K are unequal, the value of the product can be increased by replacing them by equal factors, without altering the sum. Therefore the product will be maximum when all the factors are equal. In this case each of the factors a, b, c......K will be

$$\frac{a+b+c+\ldots\ldots K}{n},\ \frac{a+b+c+\ldots\ldots K}{n},\ \ldots\ \frac{a+b+c+\ldots\ldots K}{n}$$

i. e. $\frac{S}{n},\ \frac{S}{n},\ \frac{S}{n}\ldots\ldots\frac{S}{n}$ and their sum

$=n\frac{S}{n}=S$ the given constant.

Hence the max. value of the product $(a,\ b,\ c\ldots\ldots K)$

$$\text{is}\left(\frac{S}{n}\cdot\frac{S}{n}\cdot\frac{S}{n}\ldots\ldots\frac{S}{n}\right)\text{ i. e. }\left(\frac{S}{n}\right)^n$$

∴ The Max. product $\left(\frac{S}{n}\right)^n$ is greater than the product $(abc\ldots\ldots K)$

i. e. $\left(\frac{a+b+c+\ldots\ldots K}{n}\right)^n > (abc\ldots\ldots K)$

or $\frac{a+b+c+\ldots\ldots K}{n} > (abc\ldots\ldots K)^{1/n}$

∴ Arithmetic mean of n +ive quantities is greater than their Geometric mean.

***Ex. 1 (a) Prove that** $a^2+b^2+c^2 > ab+bc+ca$ **and deduce that** $a^3+b^3+c^3 > 3abc$ **(Nagpur 34, I.F.S.30)**

(b) $(ab+bc+ca) < (a+b-c)^2+(a+c-b)^2+(a-b-c)^2$

(Nagpur 1934, Punjab 44)

We know that $\frac{a^2+b^2}{2} > \sqrt{(a^2b^2)}$ or $> ab$ ∵ AM > GM

Similarly $\frac{b^2+c^2}{2} > bc$

Remember the result of Ex. 1 and (note the procedure).

and $\frac{c^2+a^2}{2} > ca$

Adding we get $a^2+b^2+c^2 > ab+bc+ca$......(i)

Now $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$=+$ive i. e. > 0 by virtue of (i)

$\therefore\ a^3+b^3+c^3 > 3abc$

(b) On simplification it reduces to

$3(a^2+b^2+c^2)-2(ab+bc+ca) > ab+bc+ca$

or $a^2+b^2+c^2 > ab+bc+ca$ which is true

Ex. 2. $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) > 9x^2y^2z^2$

$$\frac{x^2y+y^2z+z^2x}{3} > (x^3y^3z^3)^{1/3} \text{ or } > xyz$$

and $\frac{xy^2+yz^2+zx^2}{3} > (x^3y^3z^3)^{1/3}$ or $> xyz$

Multiplying, we get

$$\frac{(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)}{9} > x^2y^2z^2$$

$\therefore\ (x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) > 9x^2y^2z^2$

***§ 6. To find the greatest value of $a^m.b^n.c^p$...... when $a+b+c$............is constant and m, n, p.........are +ive integers. (Proof) (Agra 1929)**

Since m, n, p are all constants, the expression $a^m\, b^n\, c^p$... will be max. when

$$\left(\frac{a}{m}\right)^m \left(\frac{b}{n}\right)^n \left(\frac{c}{p}\right)^p \text{.........is Max.} \qquad \text{(Note)}$$

or $\frac{a}{m} \cdot \frac{a}{m}$......$m$ factors $\times \frac{b}{n} \cdot \frac{b}{n}$.........$n$ factors

$\times \frac{c}{p} \cdot \frac{c}{p}$......$p$ factors...... is Max.

Now above is the product of $m+n+p+$......factors whose sum is $\frac{a}{m}+\frac{a}{m}+$...m factors$+\frac{b}{n}+\frac{b}{n}+$....

n factors $+\dfrac{c}{p}+\dfrac{c}{p}+\ldots\ldots p$ factors $+\ldots\ldots$

$$\text{i.e. } m\left(\frac{a}{m}\right)+n\left(\frac{b}{n}\right)+p\left(\frac{c}{p}\right)+\ldots\ldots$$

$$=a+b+c+\ldots\ldots=\text{constant (given)}$$

and hence the product will be max. when all the factors are equal

$$\text{i. e. } \frac{a}{m}=\frac{b}{n}=\frac{c}{p}=\ldots\ldots=\frac{a+b+c+\ldots\ldots}{m+n+p+\ldots\ldots}$$

$$\therefore \quad a=m.\frac{\Sigma a}{\Sigma m}, \quad b=n.\frac{\Sigma a}{\Sigma m}, \quad c=p.\frac{\Sigma a}{\Sigma m}$$

Max. value of $a^m.b^n.c^p\ldots\ldots\ldots$

$$=m^m\left(\frac{\Sigma a}{\Sigma m}\right)^m.n^n\left(\frac{\Sigma a}{\Sigma m}\right)^n.p^p\left(\frac{\Sigma a}{\Sigma m}\right)^p\ldots\ldots$$

$$=m^m.n^n.p^p\ldots\ldots\left(\frac{\Sigma a}{\Sigma m}\right)^{m+n+p+\ldots\ldots}$$

$$=m^m.n^n.p^p\ldots\ldots\left(\frac{a+b+c\ldots\ldots\ldots}{m+n+p\ldots\ldots\ldots}\right)^{m+n+p\ldots\ldots}$$

§ 7. Prove that **(Read only)**

$$\frac{p_1a_1+p_2a_2+\ldots\ldots p_na_n}{p_1+p_2+\ldots\ldots p_n} > \left[a_1^{p_1}a_2^{p_2}\ldots a_n^{p_n}\right]^{\frac{1}{p_1+p_2+\ldots p_n}}$$

Case 1. Let p_1, p_2,......p_n be all integers.

Consider p_1 numbers each equal to a_1 and p_2 numbers each equal to a_2 and so on. Applying A. M. > G. M. we get

$$\frac{(a_1+a_1+\ldots\ldots p_1 \text{ times})+(a_2+a_2+\ldots\ldots p_2 \text{ times})+\ldots\ldots}{p_1+p_2+\ldots\ldots p_n}$$

$$>[(a_1.a_1.a_1\ldots p_1 \text{ factors})(a_2.a_2.a_2\ldots p_2 \text{ factors})\ldots]^{\frac{1}{p_1+p_2+\ldots p_n}}$$

or $$\frac{p_1a_1+p_2a_2+\dots p_na_n}{p_1+p_2+\dots p_n} > \left[a_1^{p_1}a_2^{p_2}\dots a_n^{p_n}\right]^{\frac{1}{p_1+p_2+\dots p_n}}$$

Case II. If $p_1, p_2, \dots p_n$ are not integers but fractions.

Let m be the least common multiple of the denominators of $p_1, p_2, \dots\dots p_n$, then $mp_1, mp_2\dots\dots mp_n$ are all integers and we have by case I.

$$\frac{mp_1a_1+mp_2a_2+\dots\dots mp_na_n}{mp_1+mp_2+\dots mp_n}$$

$$> \left[a_1^{mp_1}a_2^{mp_2}\dots a_n^{mp_n}\right]^{\frac{1}{m(p_1+p_2+\dots p_n)}}$$

or $$\frac{p_1a_1+p_2a_2+\dots\dots+p_na_n}{p_1+p_2+\dots\dots p_n}$$

$$> \left[a_1^{p_1}.a_2^{p_2}\dots a_n^{p_n}\right]^{\frac{m}{m(p_1+p_2+\dots p_n)}}$$

or $$> \left[a_1^{p_1}.a_2^{p_2}\dots a_n^{p_n}\right]^{\frac{1}{p_1+p_2+\dots p_n}}$$

***Ex. 1.** Show that unless $p=q=r$ or $x=1$,

$$px^{q-r}+qx^{r-p}+rx^{p-q} > p+q+r.$$

Now because A. M. > G. M.

$$\frac{(x^{q-r}+x^{q-r}+\dots\dots p \text{ times})+(x^{r-p}+x^{r-p}+\dots\dots q \text{ times})+(x^{p-q}+x^{p-q}+\dots\dots r \text{ times})}{p+q+r}$$

$$>\left\{\begin{array}{l}(x^{q-r}.x^{q-r}\ldots p \text{ factors})(x^{r-p}.x^{r-p}\ldots q \text{ factors})\\ (x^{p-q}\,.\,x^{p-q}\ldots\ldots r \text{ factors})\end{array}\right\}^{\frac{1}{p+q+r}}$$

$$\text{or}\quad \frac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > \left\{ x^{p(q-r)+q(r-p)+r(p-q)} \right\}^{\frac{1}{p+q+r}}$$

$$\text{or}\quad \frac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > \left[x^{0}\right]^{\frac{1}{p+q+r}} \qquad \text{or} > 1$$

$$\therefore\quad px^{q-r}+qx^{r-p}+rx^{p-q} > p+q+r$$

In case either p=q=r or x=1, the inequality becomes equality.

*Ex. 2. Show that (Allahabad 1927)

$$\left(\frac{a+b+c\ldots\ldots k}{n}\right)^{a+b+c+\ldots\ldots k} < a^{a}\,.\,b^{b}\,.\,c^{c}\ldots\ldots k^{k}$$

Consider a numbers each equal to $\frac{1}{a}$, b numbers each equal to $\frac{1}{b}$ and c numbers each equal to $\frac{1}{c}$ and so on.

(Note)

Then we know A. M. > G. M.

$$\therefore\quad \frac{a\left(\frac{1}{a}\right)+b\left(\frac{1}{b}\right)+c\left(\frac{1}{c}\right)+\ldots\ldots\ldots k\left(\frac{1}{k}\right)}{a+b+c+\ldots\ldots k}$$

$$>\left[\left(\frac{1}{a}\right)^{a}\cdot\left(\frac{1}{b}\right)^{b}\cdot\left(\frac{1}{c}\right)^{c}\ldots\ldots\left(\frac{1}{k}\right)^{k}\right]^{\frac{1}{a+b+c+\ldots k}}$$

or $$\frac{n}{a+b+c+......k} > \left[\frac{1}{a^a \,.\, b^b \,.\, c^c..........k^k}\right]^{\frac{1}{a+b+c......k}}$$

$$\left[\frac{n}{a+b+c+......k}\right]^{a+b+c......k} > \frac{1}{a^a \,.\, b^b \,.\, c^c..... k^k}$$

Taking reciprocal and hence changing the sign of inequality, we get

$$\left[\frac{a+b+c+......k}{n}\right]^{a+b+c+......k} < a^a.b^b.c^c......k^k$$

Note :—By § 7 case 1 and 2, the above is true whether $a, b, c......k$ are integers or fractions or you may prove both the cases as in § 7.

***§ 8. If a and b are +ive and unequal then**

$$\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$$ **except when m is a +ive proper fraction.** **(Proof)** **(Agra 1946)**

$$a^m=\left(\frac{a+b}{2}+\frac{a-b}{2}\right)^m=\left(\frac{a+b}{2}\right)^m \cdot \left(1+\frac{a-b}{a+b}\right)^m$$

$$b^m=\left(\frac{a+b}{2}-\frac{a-b}{2}\right)^m=\left(\frac{a+b}{2}\right)^m\left(1-\frac{a-b}{a+b}\right)^m$$

Now $$(1+t)^m+(1-t)^m=2\left[1+\frac{m(m-1)}{2!}t^2+\frac{m(m-1)(m-2)(m-3)\,t^4}{4!}+....\right]$$

because the alternative terms cancel.

$$\therefore\quad a^m+b^m=2\left[\frac{a+b}{2}\right]^m\left[1+\frac{m(m-1)}{2!}t^2+\frac{m(m-1)(m-2)(m-3)t^4}{4!}+...\right]$$

where $t=\frac{a-b}{a+b}$

Case I. When m is a +ive integer or a negative quantity.

Suppose $m=4$, then all the three terms in the above bracket will be +ive. All other terms that follow the third term will have a factor $m-4$ and hence will vanish.

Again if $m=-$ ive quantity,

then $m(m-1)$ will be $-$ive $\times$ $-$ive $=+$ive...

Similarly all other terms in the bracket will be +ive.

$\therefore\quad a^m+b^m=2\left(\frac{a+b}{2}\right)^m$ multiplied by something +ive

but greater than 1

$$\therefore\quad \frac{a^m+b^m}{2}>\left[\frac{a+b}{2}\right]^m$$

Case II. When m is a +ive proper fraction, i.e. it lies between 0 and 1.

In this case all the terms in the bracket except the first will be $-$ive because m being positive and less than 1, $(m-1)$ will be $-$ive.

$\therefore\quad m(m-1)$ is $-$ive.

Similarly $m(m-1)(m-2)(m-3)$ will be $-$ive.

$\therefore$ It follows clearly that

$$\frac{a^m+b^m}{2}<\left(\frac{a+b}{2}\right)^m \quad \text{(Note)}$$

Case III. When m > 1.

Now when m is a +ive integer greater than 1, has been dealt in Case 1, so here we shall take up the case when m is a fraction which is greater than 1.

Put $m=\frac{1}{n}$ then n must be <1.

By Case II when n is <1.

$$\frac{A^n+B^n}{2}<\left(\frac{A+B}{2}\right)^n$$

Putting $A=a^m$, $B=b^m$, we get

$$\frac{a^{mn}+b^{mn}}{2} < \left(\frac{a^m+b^m}{2}\right)^n$$

or $$\left(\frac{a+b}{2}\right) < \left(\frac{a^m+b^m}{2}\right)^{\frac{1}{m}} \qquad \because \; mn=1$$

$$\therefore \quad \left(\frac{a+b}{2}\right)^m < \left(\frac{a^m+b^m}{2}\right)$$

Hence $$\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$$

***§ 9. If there are n +ive quantities a. b, c,.........K**

then $$\frac{a^m+b^m+c^m+\ldots..k^m}{n} > \left(\frac{a+b+c+\ldots\ldots..k}{n}\right)^m$$

except when m lies between 0 and 1 i.e. m is a +ive proper fraction. (Proof)

Or

Prove that arithmetic mean of mth powers of n quantities is greater than the arithmetic mean raised to power m except when m lies between 0 and 1.

Suppose m does not lie between 0 and 1, then consider the expression

$$a^m+b^m+c^m+\ldots\ldots k^m \qquad \text{(A)}$$

If in above we replace the unequal quantities a and b by two equal quantities $\frac{a+b}{2}$, $\frac{a+b}{2}$, causing thereby no change in sum of the numbers which will be

$$\frac{a+b}{2}+\frac{a+b}{2}+c+\ldots\ldots k$$

i. e. $a+b+c+\ldots\ldots k$ the same as before, we shall have the new form of expression A as

$$\left(\frac{a+b}{2}\right)^m+\left(\frac{a+b}{2}\right)^m+c^m+\ldots\ldots k^m\ldots\ldots(B)$$

Now $a^m+b^m > 2\left(\frac{a+b}{2}\right)^m$ Art 8.

$\therefore$ expressin A will be greater than expression B.

The effect of replacing unequal quantities of A by equal quantities is that the expression A has been diminished.

Hence so long as any two of the quantities a, b, c...k are unequal the expression A can be diminished by replacing them by equal quantities causing thereby no change in the value of a+b+c......k

Therefore the value of $a^m+b^m+c^m+\ldots\ldots k^m$ will be least when all the quantities a, b, $c\ldots\ldots k$ are equal.

In this case each of the quantities will be

$$\frac{a+b+c+\ldots\ldots k}{n} \text{ i. e. } \frac{S}{n}$$

And the least value of A will be

$$\left[\frac{S}{n}\right]^m+\left[\frac{S}{n}\right]^m+\left[\frac{S}{n}\right]^m+\ldots\ldots n \text{ times}$$

$$=n\left[\frac{S}{n}\right]^m\ldots\ldots\ldots\ldots(C)$$

Now since C is the least value of A.

$\therefore$ $A > C$

i. e. $a^m+b^m+c^m+\ldots\ldots k^m > n\left[\frac{S}{n}\right]^m$

i. e. $\frac{a^m+b^m+c^m+\ldots\ldots k^m}{n} > \left[\frac{a+b+c+\ldots\ldots k}{n}\right]^m$

Ex. 1. $16(a^3+b^3+c^3+d^3) > (a+b+c+d)^3$

Now $\frac{a^3+b^3+c^3+d^3}{4} > \left[\frac{a+b+c+d}{4}\right]^3$

$\therefore$ $16(a^3+b^3+c^3+d^3) > (a+b+c+d)^3$

***Ex. 2.** $n(n+1)^3 < 8(1^3+2^3+\ldots\ldots n^3)$

Now $\frac{1^3+2^3+\ldots\ldots n^3}{n} > \left[\frac{1+2+3+\ldots\ldots n}{n}\right]^3$

Or $> \left[\frac{n(n+1)}{2n}\right]^3$ or $> \frac{(n+1)^3}{8}$

$\therefore$ $8(1^3+2^3+\ldots\ldots n^3) > n(n+1)^3$

§10. Prove that **(Read and Result)**

$$\frac{p_1a_1{}^m+p_2a_2{}^m+\ldots\ldots p_na_n{}^m}{p_1+p_2+\ldots\ldots p_n} > \left[\frac{p_1a_1+p_2a_2+\ldots\ldots p_na_n}{p_1+p_2+\ldots\ldots p_n}\right]^m$$

except when m lies between 0 and 1

Case I. When p_1, p_2, $p_3\ldots\ldots p_n$ are integers.

Consider p_1 numbers each equal to $a_1{}^m$, p_2 numbers each equal to $a_2{}^m$ and so on.

Then by article 9 i. e. A. M. of *m*th powers is $>$ (A.M.)^m

$$\frac{(a_1{}^m+a_1{}^m+\ldots p_1 \text{ times})+(a_2{}^m+a_2{}^m+\ldots p_2 \text{ times})+\ldots}{p_1+p_2+\ldots\ldots p_n}$$

$$> \left[\frac{(a_1+a_1+\ldots p_1 \text{ times})+(a_2+a_2+\ldots p_2 \text{ times})+\ldots}{p_1+p_2+\ldots\ldots p_n}\right]^m$$

$$\frac{p_1a_1{}^m+p_2a_2{}^m+\ldots\ldots p_na_n{}^m}{p_1+p_2+\ldots\ldots p_n} > \left[\frac{p_1a_1+p_2a_2+\ldots p_na_n}{p_1+p_2+\ldots p_n}\right]^m$$

Case II. If p_1, p_2, $p_3\ldots\ldots$are not integers. Let the least common multiple of the denominators of p_1, $p_2\ldots p_n$ be M then Mp_1, Mp_2 are all integers, then we have by case I.

$$\frac{Mp_1a_1{}^m+Mp_2a_2{}^m+\ldots Mp_na_n{}^m}{Mp_1+Mp_2+\ldots Mp_n}$$

$$> \left[\frac{Mp_1a_1+Mp_2a_2+\ldots Mp_na_n}{Mp_1+Mp_2\ldots+\ldots Mp_n}\right]^m$$

or

$$\frac{p_1a_1^m+p_2a_2^m+\ldots\ldots p_na_n^m}{p_1+p_2+\ldots\ldots p_n}>\left(\frac{p_1a_1+p_2a_2+\ldots\ldots\ldots p_na_n}{p_1+p_2+\ldots\ldots p_n}\right)^m$$

MAXIMA AND MINIMA

§ 11. The articles which we have done so far can be conveniently applied for the determination of Max. and Min. Values of certain functions.

Suppose $x, y, z, \ldots w$ are n +ive$_s$ quantities and S denotes their sum and P their product ; then

Imp. 1. If x+y+z...w=S a constant then the value of xyz......w is Max. when the quantities are equal.

i. e. $x=y=z=\ldots w=\frac{S}{n}$ and Max. value becomes $\left[\frac{S}{n}\right]^n$

[Art. 4 and 5]

Imp. 2. If xyz...........w=P, a constant then the value of x+y+z+......w is least when the quantities are equal.

i. e. minimum when $x=y=z=\ldots\ldots w=P^{1/n}$

and the least value of $x+y+z\ldots\ldots w$ will be $nP^{1/n}$.

[Art. 4]

3. If x+y+z+......w=S, then the least value of

(a) $x^m+y^m+z^m+\ldots\ldots w^m$ when m does not lie between 0 and 1 occurs when $x=y=z=\ldots\ldots w=\frac{S}{n}$

and the least value becomess $n\left(\frac{S}{n}\right)^m$ [Art. 9]

(b) If x+y+z+.........w=S then greatest value of $x^m+y^m+z^m+\ldots\ldots w^m$ when m lies between 0 and 1, occurs when $x=y=z\ldots\ldots w=\frac{S}{n}$

and the greatest value becomes $n\left(\frac{S}{n}\right)^m$

4. (a) If $x^m+y^m+z^m+\ldots\ldots w^m=P$ a constant then the greatest value of $x+y+z+\ldots\ldots w$ when m does not lie between 0 and 1, occurs when $x=y=z\ldots\ldots w=\left(\frac{P}{n}\right)^{1/m}$

and the greatest value becomes $n\left(\frac{P}{n}\right)^{1/m}$

(b) If $x^m+y^m+z^m+\ldots\ldots w^m=P$, a constant, then the least value of $x+y+z\ldots\ldots w$ when m lies between 0 and 1, occurs when $x=y=z\ldots w=\left(\frac{P}{n}\right)^{1/m}$

and the least value becomes $n\left(\frac{P}{n}\right)^{1/m}$

5. **Art. No. 6 is also used in the determination of Max. and Min. of various problems.**

Ex. 1. (a) Find the Max. value of $(a+x)^3(a-x)^4$ for any real value of x numerically less than a.

Let $A=a+x$ and $B=a-x$

Then $A+B=2a$.

We have to find max. value of $A^3 \cdot B^4$

which by art. 6 will be max. when $\left(\frac{A}{3}\right)^3\left(\frac{B}{4}\right)^4$ is max.

But sum of the above factors is $3 \cdot \frac{A}{3}+4 \cdot \frac{B}{4}=A+B$

$=2a$ constant $\therefore$ Max. value will occur when

$$\frac{A}{3}=\frac{B}{4}=\frac{A+B}{7}=\frac{2a}{7} \quad \therefore A=\frac{6a}{7},\ B=\frac{8a}{7}$$

Then Max. value is $\dfrac{6^3.8^4}{7^7}.a^7$

***(b) Find the greatest value of $a^2b^3c^4$ subject to the condition $a+b+c=18$ (Agra 1942)**

***Ex. 2. Find the greatest value of x^2y^3 where $3x+2y=1$**

x^2y^3 is greatest when $\left(\frac{3x}{2}\right)^2 . \left(\frac{2y}{3}\right)^3$ is Max. (Note)

Now sum of the factors in the later product is

$$2.\frac{3x}{2}+3.\frac{2y}{3}=3x+2y=1 \text{ i. e. a given constant}$$

$\therefore$ Value will be Max. when $\dfrac{3x}{2}=\dfrac{2y}{3}=\dfrac{3x+2y}{5}=\dfrac{1}{5}$

$$x=\frac{2}{3\cdot5} \qquad y=\frac{3}{2\cdot5}$$

$\therefore$ Max. value of x^2y^3 is $\dfrac{2^2}{3^2.5^2}\cdot\dfrac{3^3}{2^3.5^3}$

$$=\frac{3}{2\cdot5^5}$$

(b) If $a^2x^4+b^2y^4=c^6$. Show that the Max. value of xy is (Madras 41, Annamalai 36)

$$\frac{c^3}{\sqrt{(2ab)}}$$

Ex, 3. Find the Min. value of $P=\dfrac{(5+x)(2+x)}{1+x}$

Putting $1+x=y$ then $x=y-1$

$$\therefore \quad P=\frac{(4+y)(1+y)}{y}=\frac{y^2+5y+4}{y}$$

$$=y+\frac{4}{y}+5=\left(y+\frac{4}{y}-4\right)+9$$

$$\text{or } \left(\sqrt{y}-\frac{2}{\sqrt{y}}\right)^2+9$$

P is Minimum when $\sqrt{y}-\frac{2}{\sqrt{y}}$ is zero

i. e. $y=2 \quad \therefore \quad x=y-1=1$

and Min. value of P then is 9

***Example 4. Prove that cube is the rectangular parallelopiped of maximum volume for given surface and of minimum surface for given volume.**

Let the three adjacent edges of a rectangular parallelopiped be x, y, z, then its surface S and volume V are given by

$$S=2(xy+yz+zx) \qquad \text{and} \qquad V=xyz$$

Putting $X=yz$, $Y=zx$, $Z=xy$

$\therefore \quad S=2(X+Y+Z)$ (given) and $V=\sqrt{(XYZ)}$

Now V is Max. when XYZ is Max.

and XYZ is Max. when $X=Y=Z$ provided

$X+Y+Z=$constant which is given to us. (11·1)

$\therefore$ For Max. volume when surface is given

$X=Y=Z$ or $yz=zx=xy$

or $\frac{1}{x}=\frac{1}{y}=\frac{1}{z}$ or $x=y=z$

i. e. all the three edges are equal.

i. e. the rectangular parallelopiped is a cube

Similarly if $V=\sqrt{(XYZ)}$ (given)

then $S=2(X+Y+Z)$ is to be Min.

Now V is given $\therefore$ XYZ is given

And we have to find the Min. value of S or $2(X+Y+Z)$

Or of $X+Y+Z$ which by 11·2 will be Min. when

$X=Y=Z$ or as above $x=y=z$

i. e. rectangular parallelopiped is a cube

Miscellaneous Theorems

***§ 12. If a and b are +ive integers, and a > b. and if x be a +ive quantity, then (Proof)**

$$\left(1+\frac{x}{a}\right)^a > \left(1+\frac{x}{b}\right)^b$$

$$\left(1+\frac{x}{a}\right)^a = 1+x+\frac{a(a-1)}{2!}\frac{x^2}{a^2}+\frac{a(a-1)(a-2)}{3!}\frac{x^3}{a^3}+\dots$$

$$=1+x+\left(1-\frac{1}{a}\right)\frac{x^2}{2!}+\left(1-\frac{1}{a}\right)\left(1-\frac{2}{a}\right)\frac{x^3}{3!}+\dots\dots\dots\text{(A)}$$

Similarly, $$\left(1+\frac{x}{b}\right)^b = 1+x+\left(1-\frac{1}{b}\right)\frac{x^2}{2!}+\left(1-\frac{1}{b}\right)\left(1-\frac{2}{b}\right)\frac{x^3}{3!}+\dots\dots\text{(B)}$$

Now A contains $a+1$ terms whereas B contains $b+1$ terms. Because $a > b$. ∴ A has greater number of terms than B and also after second term each term of A is greater than the corresponding term of B. Hence A > B.

or $$\left(1+\frac{x}{a}\right)^a > \left(1+\frac{x}{b}\right)^b$$

***Ex. 1. If α and β are +ive quantities and $\alpha > \beta$ show that**

$$\left(1+\frac{1}{\alpha}\right)^\alpha > \left(1+\frac{1}{\beta}\right)^\beta$$

(Alld. 43, Agra 49 Supp., Sagar 49)

Hence show that if n>1, then the value of $\left(1+\frac{1}{n}\right)^n$ lies between 2 and 2·718.

The 1st part follows from article 12 or we can proceed directly as in the article.

Now we know that if $\alpha>\beta$ then

$$\left(1+\frac{1}{\alpha}\right)^{\alpha} > \left(1+\frac{1}{\beta}\right)^{\beta}$$

Put $\alpha=n$ and $\beta=1$

$$\left(1+\frac{1}{n}\right)^{n} > 2 \ldots\ldots (A)$$

Again let $\alpha \to \infty$ and $\beta=n$

$$\underset{\alpha\to\infty}{\mathrm{Lt}} \left(1+\frac{1}{\alpha}\right)^{\alpha} > \left(1+\frac{1}{n}\right)^{n}$$

but $$\underset{\alpha\to\infty}{\mathrm{Lt}} \left(1+\frac{1}{\alpha}\right)^{\alpha} = e = 2{\cdot}718$$

$$\therefore \quad 2{\cdot}718 > \left(1+\frac{1}{n}\right)^{n} \ldots\ldots (B)$$

From A and B it follows that

$$2 < \left(1+\frac{1}{n}\right)^{n} < 2{\cdot}718$$

$$\therefore \quad \left(1+\frac{1}{n}\right)^{n} \text{ lies between 2 and } 2{\cdot}718$$

***§ 13. Prove that**

$$\left(\frac{1+x}{1-x}\right)^{1/x} > \left(\frac{1+y}{1-y}\right)^{1/y}$$

where both x and y are +ive proper fractions and x > y. (Agra 1947)

Proof. Let $P=\left(\frac{1+x}{1-x}\right)^{1/x}$ and $Q=\left(\frac{1+y}{1-y}\right)^{1/y}$

$$\log P = \frac{1}{x}\left[\log(1+x) - \log(1-x)\right]$$

$$= \frac{1}{x}\left[2\left(x+\frac{x^3}{3}+\frac{x^5}{5}\ldots\ldots\right)\right]$$

$$= 2\left(1+\frac{x^2}{3}+\frac{x^4}{5}+\ldots\ldots\right)$$

Similarly $\log Q=2\left(1+\frac{y^2}{3}+\frac{y^4}{5}+\dots\right)$.

now x being $> y$ $\therefore$ $\log P > \log Q$ i. e. $P > Q$

or $\left(\frac{1+x}{1-x}\right)^{1/x} > \left(\frac{1+y}{1-y}\right)^{1/y}$

***Ex. 1. If a, b, c are in descending order of magnitude**

show that $\left(\frac{a+c}{a-c}\right)^a < \left(\frac{b+c}{b-c}\right)^b$ **(Rajputana 50)**

we are given that $a > b > c$ therefore

both $\frac{c}{a}$ and $\frac{c}{b}$ are +ive proper fractions and $\frac{c}{a}$ is less

then $\frac{c}{b}$. Let us denote them by y and x respectively.

We have to prove that $\left(\frac{a+c}{a-c}\right)^a < \left(\frac{b+c}{b-c}\right)^b$

$$\text{or} \quad \left\{\frac{1+\frac{c}{a}}{1-\frac{c}{a}}\right\}^a < \left\{\frac{1+\frac{c}{b}}{1-\frac{c}{b}}\right\}^b$$

$$\text{or} \quad \left\{\frac{1+\frac{c}{a}}{1-\frac{c}{a}}\right\}^{a/c} < \left\{\frac{1+\frac{c}{b}}{1-\frac{c}{b}}\right\}^{b/c}$$

$$\text{or} \quad \left(\frac{1+y}{1-y}\right)^{1/y} < \left(\frac{1+x}{1-x}\right)^{1/x}$$

or $\left(\frac{1+x}{1-x}\right)^{1/x} > \left(\frac{1+y}{1-y}\right)^{1/y}$

Where x and y are both +ive proper fractions and $x > y$.

This we have proved above in article 13 or may be proved here again:

***§14. Prove** $(1+x)^{1+x}(1-x)^{1-x} > 1$ if x is a +ive proper fraction and deduce that $a^a \cdot b^b > \left(\frac{a+b}{2}\right)^{a+b}$

(Agra 48, Rajputana 48)

If P > 1 then log P must be +ive. (Proof)

Let $P=(1+x)^{1+x}(1-x)^{1-x}$

then $\log P=(1+x)\log(1+x)+(1-x)\log(1-x)$

$$,, =\{\log(1+x)+\log(1-x)\}+x\{\log(1+x)-\log(1-x)\}$$

$$=2\left(-\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^6}{6}\ldots\ldots\right)+2x\left(x+\frac{x^3}{3}+\frac{x^5}{5}\ldots\ldots\right)$$

$$=2\left\{\left(-\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^6}{6}\ldots\right)+\left(\frac{x^2}{1}+\frac{x^4}{3}+\frac{x^6}{5}+\ldots\right)\right\}$$

$$=2\left\{\left(\frac{x^2}{1}-\frac{x^2}{2}\right)+\left(\frac{x^4}{3}-\frac{x^4}{4}\right)+\left(\frac{x^6}{5}-\frac{x^6}{6}\right)+\ldots\right\}$$

$$=2\left(\frac{x^2}{1\cdot2}+\frac{x^4}{3\cdot4}+\frac{x^6}{5\cdot6}\ldots\ldots\right)$$

which is definitely a +ive quantity

log P being +ive quantity ∴ $P > 1$

Deduction :—$a^a b^b > \left(\frac{a+b}{2}\right)^{a+b}$

Putting $x=\frac{z}{u}$ where $u > z$ because $x < 1$ then

$$\left(1+\frac{z}{u}\right)^{1+\frac{z}{u}}\left(1-\frac{z}{u}\right)^{1-\frac{z}{u}} > 1$$

or $\left(\frac{u+z}{u}\right)^{\frac{u+z}{u}}\left(\frac{u-z}{u}\right)^{\frac{u-z}{u}} > 1$

or $\left(\frac{u+z}{u}\right)^{u+z}\left(\frac{u-z}{u}\right)^{u-z} > 1^{u}$ or 1

or $(u+z)^{u+z}(u-z)^{u-z} > u^{u+z+u-z}$ or $> u^{2u}$

Put $u+z=a$ and $u-z=b$, then $u=\frac{a+b}{2}$

$\therefore \quad a^{a}.b^{b} > \left(\frac{a+b}{2}\right)^{a+b}$

***§ 15. Prove that $(1+x)^{1-x}(1-x)^{1+x} < 1$ where x is a positive proper fraction and deduce that**

$$a^{b}b^{a} < \left(\frac{a+b}{2}\right)^{a+b}$$ **(Punjab 50)**

Proof—Proceed exactly like article 14 and remember that if P is < 1, then log P must be $-$ive.

***§ 16. Prove independently that**

$$a^{a}.b^{b} > \left(\frac{a+b}{2}\right)^{a+b} > a^{b}.b^{a}$$ **(Agra 43)**

Consider a quantities each equal to $\frac{1}{a}$, b quantities each equal to $\frac{1}{b}$ then since A. M. > G. M. (Note)

$$\frac{\left[\frac{1}{a}+\frac{1}{a}+\frac{1}{a}\ldots a \text{ times}\right]+\left[\frac{1}{b}+\frac{1}{b}+\frac{1}{b}\ldots b \text{ times}\right]}{a+b}$$

$$> \left\{\left(\frac{1}{a}\cdot\frac{1}{a}\cdot\frac{1}{a}\ldots\ldots a \text{ factors}\right)\left(\frac{1}{b}\cdot\frac{1}{b}\cdot\frac{1}{b}\ldots\ldots b \text{ factors}\right)\right\}^{\frac{1}{a+b}}$$

or $$\frac{a.\frac{1}{a}+b.\frac{1}{b}}{a+b} > \left\{\frac{1}{a^a}.\frac{1}{b^b}\right\}^{\frac{1}{a+b}}$$

or $$\left\{\frac{2}{a+b}\right\} > \left\{\frac{1}{a^a.b^b}\right\}^{\frac{1}{a+b}}$$

or $$\left\{\frac{2}{a+b}\right\}^{a+b} > \frac{1}{a^a.b^b}$$

Taking reciprocal and hence changing the sign of inequality, we get

$$\left\{\frac{a+b}{2}\right\}^{a+b} < a^a.b^b$$

or $$a^a.b^b > \left\{\frac{a+b}{2}\right\}^{a+b}$$

Note—a and b can be integral as well as fractional. Refer article 7.

11. Consider b quantities each equal to a and a quantities each equal to b. Then since A. M. > G. M (Note)

$$\frac{(a+a+a+\ldots\ldots b \text{ times})+(b+b+b+\ldots\ldots a \text{ times})}{a+b}$$

$$> [(a.a.a\ldots\ldots b \text{ factors})(b.b.b\ldots\ldots a \text{ factors})^{\frac{1}{a+b}}$$

or $$\frac{ab+ab}{a+b} > (a^b.b^a)^{\frac{1}{a+b}}$$

or $$\frac{2ab}{a+b} > (a^b.b^a)^{\frac{1}{a+b}}$$

But $\frac{a+b}{2}$ being A. M. of a and b is greater than $\frac{2ab}{a+b}$ their H. M.

$$\therefore \quad \frac{a+b}{2} > \frac{2ab}{a+b} > [a^b . b^a]^{\frac{1}{a+b}}$$

$$\text{or} \quad \left[\frac{a+b}{2}\right]^{a+b} > a^b . b^a$$

$$\therefore \quad a^a . b^b > \left\{\frac{a+b}{2}\right\}^{a+b} > a^b . b^a$$

§17. If a_1, a_2 are +ive numbers (including zero) and p, q, r......are +ive integers then prove.

$$\frac{a_1^{p+q}+a_2^{p+q}}{2} \geqslant \left[\frac{a_1^p+a_2^p}{2}\right]\left[\frac{a_1^q+a_2^q}{2}\right] \text{......(A)}$$

$$\text{and} \quad \frac{a_1^{p+q+r+\cdots}+a_2^{p+q+r+\cdots}}{2} \geqslant \left(\frac{a_1^p+a_2^p}{2}\right)\left(\frac{a_1^q+a_2^q}{2}\right)\left(\frac{a_1^r+a_2^r}{2}\right) \text{.........(B)}$$

$$\frac{a_1^m+a_2^m}{2} \geqslant \left(\frac{a_1+a_2}{2}\right)^m \text{.........(C)}$$

Proof.

Now $a_1^p - a_2^p$ and $a_1^q - a_2^q$ have the same sign or each of them is equal to zero, therefore

$$(a_1^p - a_2^p)(a_1^q - a_2^q) \geqslant 0$$

$$\text{or} \quad a_1^{p+q}+a_2^{p+q} \geqslant a_1^p a_2^q + a_1^q a_2^p$$

$$\therefore \quad 2(a_1^{p+q}+a_2^{p+q}) \geqslant a_1^{p+q}+a_2^{p+q}+a_1^p a_2^q+a_1^q a_2^p$$

$$\text{or} \quad \geqslant (a_1^p+a_2^p)(a_1^q+a_2^q)$$

Dividing by 4 we get

$$\frac{a_1^{p+q}+a_2^{p+q}}{2} \geqslant \left(\frac{a_1^p+a_2^p}{2}\right)\left(\frac{a_1^q+a_2^q}{2}\right) \text{ which is A}$$

Again by A

$$\frac{a_1^{p+r}+a_2^{p+r}}{2} \geqslant \left(\frac{a_1^p+a_2^p}{2}\right)\left(\frac{a_1^r+a_2^r}{2}\right)$$

---

§17, 18 and 19 may be left on 1st reading.

Put $P=p+q$.

$$\frac{a_1^{p+q+r}+a_2^{p+q+r}}{2} \geqslant \left(\frac{a_1^{p+q}+a_2^{p+q}}{2}\right)\left(\frac{a_1^{r}+a_2^{r}}{2}\right)$$

$$\text{or} \quad \geqslant \left\{\frac{a_1^{p}+a_2^{p}}{2}\right\}\left\{\frac{a_1^{q}+a_2^{q}}{2}\right\}\left\{\frac{a_1^{r}+a_2^{r}}{2}\right\}$$

By repeated application of above we get the result B.

Putting $p=q=r=\ldots\ldots=1$ in B, and supposing that $p+q+r+\ldots=m$, we get

$$\frac{a_1^m+a_2^m}{2} \geqslant \left(\frac{a_1+a_2}{2}\right)\left(\frac{a_1+a_2}{2}\right) \ldots\ldots\ldots m \text{ factors}$$

$$\text{i. e. } \frac{a_1^m+a_2^m}{2} \geqslant \left(\frac{a_1+a_2}{2}\right)^m \ldots\ldots\ldots \text{C}$$ ✓✓

§ 18. Generalized form of § 17

$$\frac{a_1^{p+q}+a_2^{p+q}+\ldots\ldots a_n^{p+q}}{n} \geqslant \left(\frac{a_1^{p}+a_2^{p}+\ldots\ldots a_n^{p}}{n}\right)\left(\frac{a_1^{q}+a_2^{q}+\ldots\ldots a_n^{q}}{n}\right)\ldots\ldots\ldots \text{A}$$

$$\frac{a_1^{p+q+r+\cdot}+a_2^{p+q+r+\cdot}+\ldots a_n^{p+q+r+\cdot}}{n} \geqslant \left(\frac{a_1^{p}+a_2^{p}+\ldots\ldots a_n^{p}}{n}\right)\left(\frac{a_1^{q}+a_2^{q}+\ldots a_n^{q}}{n}\right)\left(\frac{a_1^{r}+a_2^{r}+\ldots a_n^{r}}{n}\right)\ldots\ldots \text{B}$$

Put $p=q=r=\ldots=1$ and supposing $p+q+r\ldots=m$ in B

$$\frac{a_1^m+a_2^m+\ldots a_n^m}{n} > \left(\frac{a_1+a_2+\ldots a_n}{n}\right)^m \ldots\ldots\ldots \text{C}$$

§ 19, Weirstrass inequality

If $a_1, a_2, \ldots a_n$ are positive numbers less than unity and $S_n=a_1+a_2+\ldots a_n$ then

$$1-S_n<(1-a_1)(1-a_2)\ldots(1-a_n)<\frac{1}{1+S_n}\ldots \text{A}$$

$$1+S_n<(1+a_1)(1+a_2)\ldots(1+a_n)<\frac{1}{1-S_n}\ldots \text{B}$$

where in the latter inequality, it is supposed that $S_n < 1$.

Proof. $(1-a_1)(1-a_2)=1-(a_1+a_2)+a_1a_2$

$\therefore \qquad >1-(a_1+a_2)\ldots\ldots(1)$

$(1-a_1)(1-a_2)(1-a_3)=\{1-(a_1+a_2)+a_1a_2\}(1-a_3)$

$\therefore \qquad >\{1-(a_1+a_2)\}(1-a_3)$

$\therefore \qquad >1-(a_1+a_2+a_3)$ (by 1)......(2)

Proceeding in this way, we get

$(1-a_1)(1-a_2)(1-a_3)\ldots(1-a_n)>1-(a_1+a_2+a_3\ldots a_n)$

or $\qquad >1-S_n$

$\therefore \quad 1-S_n<(1-a_1)(1-a_2)(1-a_3)\ldots\ldots(1-a_n)$

which proves the 1st part of inequality (A)

Again $(1+a_1)(1+a_2)=1+(a_1+a_2)+a_1a_2$

$\therefore \qquad >1+(a_1+a_2)\ldots\ldots(3)$

$(1+a_1)(1+a_2)(1+a_3)=\{1+(a_1+a_2)+a_1a_2\}(1+a_3)$

$>[1+(a_1+a_2)](1+a_3)$

$>1+(a_1+a_2+a_3)$ by (3)......(4)

Proceeding in this way, we get

$(1+a_1)(1+a_2)(1+a_3)\ldots\ldots(1+a_n)>1+(a_1+a_2+\ldots\ldots a_n)$

or $\qquad >1+S_n$

$\therefore \quad 1+S_n<(1+a_1)(1+a_2)\ldots\ldots(1+a_n)$

which proves the 1st part of inequality B.

Now $a_1, a_2, \ldots\ldots a_n$ are all positive integers less than 1

$\therefore \quad 1-a_1^2<1$ or $(1-a_1)(1+a_1)<1$

or $(1-a_1)<\dfrac{1}{1+a_1}$

$\therefore \quad (1-a_1)(1-a_2)(1-a_3)\ldots\ldots(1-a_n)$

$$<\frac{1}{(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n)}\ldots\ldots(5)$$

Now $(1+a)(1+a_2)\ldots\ldots(1+a_n)>1+S_n$

by 1st part of inequality B.

Taking reciprocal and changing the sign of inequality, we get $\dfrac{1}{(1+a_1)(1+a_2)\ldots(1+a_n)}<\dfrac{1}{1+S_n}$

Hence from (5) we get

$$(1-a_1)(1-a_2)\ldots\ldots(1-a_n)<\frac{1}{1+S_n}$$

which proves the second part of inequality No. A.

Similarly $(1+a_1)(1+a_2)\ldots\ldots(1+a_n)$

$$<\frac{1}{(1-a_1)(1-a_2)\ldots\ldots(1-a_n)}\ldots\ldots(6)$$

Now $(1-a_1)(1-a_2)\ldots\ldots(1-a_n)>1-S_n$

by 1st part of inequality (A)

Taking reciprocal and changing the sign of inequality, we get $\dfrac{1}{(1-a_1)(1-a_2)\ldots(1-a_n)}<\dfrac{1}{1-S_n}$

Hence from (6) we get

$$(1+a_1)(1+a_2)\ldots(1+a_n)<\frac{1}{1-S_n}$$

which proves the 2nd part of inequality No. B.

EXERCISE 3

1. Prove the following :—

(a) $(ab+xy)(ax+by)>4axby$

Now $ab+xy>2\sqrt{(abxy)}$ ✓ A. M.>G. M.

and $ax+by>2\sqrt{(abxy)}$

Multiplying $(ab+xy)(ax+by)>4abxy$ ✓

(b) $(b+c)(c+a)(a+b)>8abc$ Procced as in (a)

(c) $(a+b+c)(bc+ca+ab)>9abc$ Proceed as in (a)

(d) $a^2cd+b^2ad+c^2ab+d^2cb>4abcd$ ✓ Apply A.M.>G.M

(e) $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}+\dfrac{d}{a}>4$ ✓ Apply A.M.>G.M.

(f) $\left(\frac{a}{e}+\frac{b}{f}+\frac{c}{g}\right)\left(\frac{e}{a}+\frac{f}{b}+\frac{g}{c}\right)>9$

Proceed as in part (a)

*(g) If sum of any two of the quantities x, y, z be greater than the third, then prove that

(a) $(x+y+z)^3>27(y+z-x)(z+x-y)(x+y-z)$

(Rajputana 1952)

and (b) $xyz>(y+z-x)(z+x-y)(x+y-z)$

(Agra 1929, 39, Madras 41)

Now $x+y>z$ given. $\therefore$ $x+y-z$ is +ive.

Similarly all other factors are +ive.

Put $y+z-x=a$, $z=\frac{a+b}{2}$

$z+x-y=b$ $x=\frac{b+c}{2}$

$x+y-z=c$ $y=\frac{a+c}{2}$ $\therefore$ $x+y+z=a+b+c$

Substituting in (a), we have to prove

(a) $(a+b+c)^3>27abc$

But we know that $(a+b+c)>3(abc)^{1/3}$. Hence on taking cube of both sides, we get the required result.

(b) Substituting in (b) we have to prove

$\frac{a+b}{2}\cdot\frac{b+c}{2}\cdot\frac{c+a}{2}>abc$ which is clearly true.

*(g$_1$) If the sum of any three of the quantities a, b, c, d be greater than twice the fourth, then prove that

$$abcd>(b+c+d-2a)(c+d+a-2b)(d+a+b-2c)\times(a+b+c-2d)$$

Now since $b+c+d>2a$ (given)

$\therefore$ $b+c+d-2a$ is +ive

Similarly all other factors are +ive.

Put $b+c+d-2a=x$......(1)

$c+d+a-2b=y$......(2)

$d+a+b-2c=z$......(3)

$a+b+c-2d=w$......(4)

Adding (2), (3), (4), we get $a=\frac{y+z+w}{3}$

Similarly $b=\frac{z+w+x}{3}$

$c=\frac{w+x+y}{3}$

Substituting above, $d=\frac{x+y+z}{3}$

∴ We are to prove that

$$\left(\frac{y+z+w}{3}\right)\left(\frac{z+w+x}{3}\right)\left(\frac{w+x+y}{3}\right)\left(\frac{x+y+z}{3}\right)>xyzw$$

Now applying A. M.>G. M. on each bracket of L. H. S. and multiplying, we get the required result.

(h) $6abc<bc(b+c)+ca(c+a)+ab(a+b)$

In other words, we have to prove that

$b^2c+c^2b+c^2a+a^2c+a^2b+ab^2>6abc$. Now A. M.>G.M.

(h$_1$) $6abc<a^2(1+b^2)+b^2(1+c^2)+c^2(1+a^2)$

(Proceed as in h)

(i) $b^2c^2+c^2a^2+a^2b^2>abc(a+b+c)$ **(Punjab 49)**

$b^2c^2+c^2a^2>2(a^2b^2c^4)^{1/2}$ or $>2abc^2$

Similarly $c^2a^2+a^2b^2>2bca^2$ and $a^2b^2+b^2c^2>2cab^2$

Add all the three inequalities

*(j) $\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

$$\frac{bc}{a^3}+\frac{ca}{b^3}>2\left(\frac{bac^2}{a^3b^3}\right)^{1/2} \text{ or } >\frac{2c}{ab}$$

Similarly writing other inequalities and adding, we get

$$\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}>\frac{c}{ab}+\frac{b}{ca}+\frac{a}{bc}\text{......(A)}$$

Again $\frac{c}{ab}+\frac{b}{ca}>2\left[\frac{bc}{bc.a^2}\right]^{\frac{1}{2}}$ or $>2.\frac{1}{a}$

Similarly writing other inequalities, and adding we get

$\frac{c}{ab}+\frac{b}{ca}+\frac{a}{bc}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ ....B

$\therefore\ \frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ from A and B

(k) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>\frac{1}{\sqrt{(bc)}}+\frac{1}{\sqrt{(ca)}}+\frac{1}{\sqrt{(ab)}}$

$\frac{1}{b}+\frac{1}{c}>2\left[\frac{1}{b}\cdot\frac{1}{c}\right]^{\frac{1}{2}}$ or $>2\frac{1}{\sqrt{(bc)}}$ and so on

**(l) If $x+y+z=1$

(Agra 40, 43, Allahabad 37, 40, Travancore 42: Raj. 51. Pb. 50)

then $(1-x)(1-y)(1-z)>8xyz$

$1-x=y+z>2(yz)^{1/2}$ Similarly write other inequalities for other brackets and multiply

**(m) If $a+b+c=1$ then

$$\frac{8}{27abc}>\left\{\frac{1}{a}-1\right\}\left\{\frac{1}{b}-1\right\}\left\{\frac{1}{c}-1\right\}>8$$

On multiplying by abc we have to prove that

$\frac{8}{27}>(1-a)(1-b)(1-c)>8abc$

Now $\frac{(1-a)+(1-b)+(1-c)}{3}>\{(1-a)(1-b)(1-c)\}^{1/3}$

or $\frac{3-(a+b+c)}{3}>\{(1-a)(1-b)(1-c)\}^{1/3}$

or $(\frac{2}{3})^3>(1-a)(1-b)(1-c)$ which proves the first part.

The second part of the inequality follows from part (l)

*(n) $2(a^3+b^3+c^3)>bc(b+c)+ca(c+a)+ab(a+b)$ ✓

The later expression can also be put as

$$a^2(b+c)+b^2(c+a)+c^2(a+b)$$

or $$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$$

We know that $a^2+b^2>2ab$

∴ $a^2+b^2-ab>ab$

Multiplying both sides by $(a+b)$ we get

$$(a+b)(a^2-ab+b^2)>ab(a+b)$$

or $$a^3+b^3>ab(a+b)$$

Similarly $$b^3+c^3>bc(b+c)$$

$$c^3+a^3>ca(c+a)$$

Adding we get

$$2(a^3+b^3+c^3)>ab(a+b)+bc(b+c)+ca(c+a)$$

*(o) $3(a^3+b^3+c^3)>(a+b+c)(a^2+b^2+c^2)$

We have to prove $3(a^3+b^3+c^3)>a^3+b^3+c^3+a(b^2+c^2)$
$+b(c^2+a^2)+c(a^2+b^2)$

or $2(a^3+b^3+c^3)>a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$

which is proved above

The question can also be written as

$$\frac{a^3+b^3+c^3}{3}>\frac{a+b+c}{3}\cdot\frac{a^2+b^2+c^2}{3}$$

(p) If a, b, c, are +ive then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}>\frac{3}{2}\quad\text{(Agra 37)}$$

or $2\{a(a+c)(a+b)\}+\ldots+\ldots>3(a+b)(b+c)(c+a)$

or $2\{a^3+a^2(b+c)+abc\}+\ldots+\ldots$
$>6abc+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)$

or $2(a^3+b^3+c^3)+2a^2(b+c)+2b^2(c+a)+2c^2(a+b)+6abc$
$>6abc+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)$

or $2(a^3+b^3+c^3)>a^2(b+c)+b^2(c+a)+c^2(a+b)$

which we have proved above.

(q) $n^n>1.3.5\ldots\ldots(2n-1)$ **(Agra 47 Supp.)**

Apply A. M. of 1, 3, 5......$(2n-1)$ i.e. of n numbers is greater than their G. M.

and remember that $1+3+5$......upto n terms

$$=\frac{n}{2}\{2a+(n-1)d\} \text{ where } a=1 \text{ and } d=2$$

*(q) $(n!)^2>n^n$. (Travan. 45, Luck. 46 Supp., Pb. 42)

We know, $n-r>\frac{n-r}{r}$ where r is a +ive integer

$\therefore \quad r(n-r)>n-r$

or $\quad r(n-r+1)>n$

Put $r=1, 2, 3......n$.

$1\,.\,n=n$ a case of equality

$2\,.\,(n-1)>n$

$3\,.\,(n-2)>n$

... ...

$(n-1)\,.\,2>n$

$n\,.\,1=n$

Multiplying, we get

$$(n!)(n!)>n^n$$

or $$(n!)^2>n^n$$

Alternative method.

Let us consider A. M.>G. M. of $(n-1)$ quantities.

$$\frac{1}{1.2}, \frac{1}{2.3}, \ldots\ldots \frac{1}{(n-1)n}$$

$$\therefore \frac{\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\frac{1}{(n-)n}}{n-1}>\left\{\frac{1}{(1.2)(2.3)(3.4)\ldots[(n-1)n]}\right\}^{\frac{1}{n-1}}$$

or $$\frac{(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots\ldots\left(\frac{1}{n-1}-\frac{1}{n}\right)}{n-1}$$

$$>\left[\frac{1}{1^2\,.\,2^2\,.\,3^2\ldots\ldots(n-1)^2\,.\,n}\right]^{\frac{1}{n-1}}$$

or $$\frac{\left(1-\frac{1}{n}\right)}{n-1} > \left[\frac{1}{[(n-1)!]^2 . n}\right]^{\frac{1}{n-1}}$$

or $$\frac{1}{n} > \left[\frac{1}{[(n-1)!]^2 . n}\right]^{\frac{1}{n-1}}$$

or $\left(\frac{1}{n}\right)^{n-1} > \frac{1}{[(n-1)!]^2 . n}$. Taking reciprocal

∴ $n^{n-1} < [(n-1)!]^2 . n$

Multiplying both sides by n, we get

$$n^n < [(n-1)!]^2 . n^2 \quad \text{or} \quad < (n!)^2$$

∴ $(n!)^2 > n^n$

*(r) $2 . 4 . 6 ...... 2n < (n+1)^n$ (Nagpur 38, Luck. 49)

Proceed as in (q) part.

(s) $(1^r+2^r+3^r+......n^r)^n > n^n(n!)^r$ (Pb. 51)

Now $\frac{1^r+2^r+3^r+......n^r}{n} > [1^r . 2^r . 3^r ..... n^r]^{1/n}$

or $(1^r+2^r+3^r......n^r)^n > n^n(n!)^r$

*(t) $(n!)^3 < n^n\left(\frac{n+1}{2}\right)^{2n}$ (Pb. 51)

Apply A.M. of $1^3+2^3+......n^3$ is greater than their G.M. and remember that $1^3+2^3+......n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

**(u) $2^n > 1+n\sqrt{(2^{n-1})}$

(Agra 33, Luck. 42, 44 Supp., Madras 42, I. C. S. 43, Travancore 47, Benares 49, Raj. 53)

or $$\frac{2^n-1}{2-1} > n . 2^{\frac{n-1}{2}}$$

But by algebra, we know that $\frac{a(r^n-1)}{r-1}$ is the sum of a G. P. whose first term is a and common ratio is r.

In the above $a=1$ and $r=2$ $\therefore$ we have to prove that

$$1+2+2^2+2^3+\ldots\ldots 2^{n-1} > n \,.\, 2^{\frac{n-1}{2}}$$

Now $\frac{1+2+2^2+2^3+\ldots\ldots 2^{n-1}}{n}$ is

$$>(1\,.\,2\,.\,2^2\,.\,2^3\ldots\ldots 2^{n-1})^{1/n} \text{ or } >(2^{1+2+3+\cdots n-1})^{1/n}$$

or $>[2^{\frac{(n-1)n}{2}}]^{\frac{1}{n}}$ or $>2^{\frac{n-1}{2}}$.

$$\therefore\ 1+2+2^2+\ldots\ldots 2^{n-1} > n . 2^{\frac{n-1}{2}}.$$

(v) $(1+x^3)(1+y^3)(1+z^3) > (1+xyz)^3$

(Agra 35, Luck. 43, Allahabad 44)

On simplification we have to prove that

$$1+(x^3+y^3+z^3)+(x^3y^3+y^3z^3+z^3x^3)+x^3y^3z^3$$
$$>1+3xyz+3x^2y^2z^2+x^3y^3z^3$$

Now apply on each bracket in L.H.S. that A.M.>G.M.

*(w) $\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}<\frac{1}{2}(a+b+c)$

$$(b+c)>2\sqrt{(bc)}$$

Square both sides

$\therefore$ $4bc<(b+c)^2$

or $\frac{4bc}{b+c}<(b+c)$

Similarly write other similar inequalities and add.

*(x) Prove that

$$abcd>81(s-a)(s-b)(s-c)(s-d)$$

where $3s=a+b+c+d$

$$a=3s-b-c-d \text{ or } =(s-d)+(s-b)+(s-c)$$
$$>3\{(s-b)(s-c)(s-d)\}^{1/3} \quad \because \text{A. M.}>\text{G M.}$$

Similarly write other inequalities for b, c and d and then multiply.

(x_1) $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\dots\dots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}>n$ (Utkal 47)

Apply A. M.>G. M.

***(y) Prove that**

$$a_1a_2\dots\dots a_n>(n-1)^n(s-a_1)(s-a_2)\dots\dots(s-a_n)$$

where $(n-1)s=a_1+a_2+a_3+\dots\dots a_n$.

Above is a generalised form of X part.

$$\begin{aligned} a_1&=(n-1)s-(a_2+a_3+\dots\dots a_n)\\ &=(s-a_2)+(s-a_3)+\dots\dots(s-a_n)\\ &>(n-1)\{(s-a_2)(s-a_3)\dots\dots(s-a_n)\}^{\frac{1}{n-1}} \end{aligned}$$

Similarly write n inequalities of the above form and multiply. Remember that the factor $(s-a_1)$ on the R. H. S. will occur $(n-1)$ times and similarly other factors.

$$\therefore\ a_1a_2a_3\dots\dots a_n>(n-1)^n\{(s-a_1)^{n-1}(s-a_2)^{n-1}\dots\dots(s-a_n)^{n-1}\}^{\frac{1}{n-1}}$$

or $$>(n-1)^n(s-a_1)(s-a_2)\dots\dots(s-a_n)$$

***(z) If $s=a+b+c+d$, prove that**

$$(s-a)(s-b)(s-c)(s-d)>81abcd \qquad \text{(Raj. 49)}$$

$$s-a=b+c+d>3(bcd)^{1/3}$$

Similarly write similar inequalities for $(s-b)$, $(s-c)$ and $(s-d)$ and multiply.

2. Prove that

***(a)** $$\left[\frac{x^2+y^2+z^2}{x+y+z}\right]^{x+y+z}>x^x.y^y.z^z>\left[\frac{x+y+z}{3}\right]^{x+y+z}$$

(Allahabad 48, Agra 38, Pb. 49)

Consider x quantities each equal to x and y quantities each equal to y and z quantities each equal to z, then since A. M.>G. M., we have

$$\frac{(x+x+x\ldots x\text{ times})+(y+y+y\ldots y\text{ times})+(z+z+z\ldots z\text{ times})}{x+y+z}$$

$$>\{(x.x.x\ldots x\text{ factors})(y.y.y\ldots y\text{ factors})(z.z.z\ldots z\text{ factors})\}^{\frac{1}{x+y+z}}$$

or $\dfrac{x^2+y^2+z^2}{x+y+z}>[x^x.y^y.z^z]^{\frac{1}{x+y+z}}$

$\because$ $x+x+x\ldots x\text{ times}=x.x=x^2.$

and $x.x.x\ldots\ldots x\text{ factors}=x^x$

or $\left[\dfrac{x^2+y^2+z^2}{x+y+z}\right]^{x+y+z}>x^xy^yz^z$

which proves the first part.

Now proceeding as in example 2.of page 101, we get

$$x^x.y^y.z^z>\left[\frac{x+y+z}{3}\right]^{x+y+z}$$

which proves the second part.

Note—The students should prove the 2nd part by the method of ex. 2 of page 101 independently.

*(b) $\left[\dfrac{bc+ca+ab}{a+b+c}\right]^{a+b+c}>\sqrt{[(bc)^a(ca)^b(ab)^c]}$

or $\left[\dfrac{bc+ca+ab}{a+b+c}\right]^{2(a+b+c)}>(a^{b+c}b^{c+a}c^{a+b})$

From above it is clear that we must consider $2(a+b+c)$ numbers and $(b+c)$ should be equal to a, $(c+a)$ equal to b and $(a+b)$ equal to c.

$\because$ A M.>G. M., we have

$$\frac{[a+a+\ldots(b+c)\text{ times}]+[b+b+\ldots(c+a)\text{ times}]+[c+c+\ldots(a+b)\text{ times}]}{b+c+c+a+a+b}$$

$$>[a^{b+c}b^{c+a}c^{a+b}]^{\frac{1}{2(a+b+c)}}$$

or $$\frac{a(b+c)+b(c+a)+c(a+b)}{2(a+b+c)}>\left(a^{b+c}b^{c+a}c^{a+b}\right)^{\frac{1}{2(a+b+c)}}$$

or $$\left(\frac{ab+bc+ca}{a+b+c}\right)^{2(a+b+c)}>a^{b+c}b^{c+a}c^{a+b}$$

or $$>(bc)^a(ca)^b(ab)^c$$

$$\therefore\quad\left(\frac{ab+bc+ca}{a+b+c}\right)^{a+b+c}>\sqrt{[(bc)^a(ca)^b(ab)^c]}$$

3. (a) $8(a^2+b^2)(a^3+b^3)>(a+b)^5$

Apply A. M. of mth powers $>$(A. M.)^m on each bracket of L. H. S.

*(b) $n(n+1)^3<8(1^3+2^3+\ldots\ldots n^3)$

(Benares 30, Nagpur 38)

Apply A. M. of $1^3, 2^3\ldots\ldots n^3$ is $>$ cube of A. M. of 1, 2, 3,……n.

***(c) If a, b, c are in H.P. and n does not lie between 0 and 1, then $a^n+c^n>2b^n$. (Luck. 46, Agra 46, 51, 53)**

$$\frac{a^n+c^n}{2}>\left(\frac{a+c}{2}\right)^n$$

But we know that A. M.>G. M.>H. M.

$\therefore\ \frac{a+c}{2}$ the A. M. of a and c is $>$ H. M. of a and c i.e. b

$\because\ a, b, c$ are in H. P.

$$\therefore\ \left(\frac{a+c}{2}\right)^n>b^n$$

$$\therefore\ \frac{a^n+c^n}{2}>\left(\frac{a+c}{2}\right)^n>b^n\quad\therefore\ a^n+c^n>2b^n$$

(d) Show that sum of the mth powers of n even numbers is $>n(n+1)^m$ if $m>1$.

$$\frac{2^m+4^m+\ldots\ldots(2n)^m}{n}>\left(\frac{2+4+6\ldots\ldots2n}{n}\right)^m$$

or $\quad 2^m+4^m+\ldots\ldots(2n)^m>n\left[\frac{2.n(n+1)}{2n}\right]^m.$

or $\quad >n(n+1)^m$

*(c) If $x+y+z=1$ then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}>9$

(Agra 40, 44, Rajputana 51, Sagar 48)

$$\frac{x^{-1}+y^{-1}+z^{-1}}{3}>\left[\frac{x+y+z}{3}\right]^{-1}. \text{ or } >\left[\frac{1}{3}\right]^{-1} \text{ or } >3$$

$\therefore \quad x^{-1}+y^{-1}+z^{-1}>9$

or $\quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}>9$

$\frac{x^{-1}+y^{-1}+z^{-1}}{3} > \left(\frac{x+y+}{3}\right.$

$\therefore x^{-1}+y^{-1}+z^{-1}$

*(e) Find the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ for +ive values of x, y, z which satisfy the condition $x+y+z=c$.

(Agra 54) (Ans. $9c$)

*(f) If a, b, c be unequal +ive quantities such that the sum of any two is greater than the third, then

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{9}{a+b+c}$$

Now $\quad b+c>a \qquad \therefore \quad b+c-a=+\text{ive}$

Similarly all other factors are +ive.

Put $\quad b+c-a=x\ldots\ldots(1)$

$c+a-b=y\ldots\ldots(2)$

$a+b-c=z\ldots\ldots(3)$

Adding $\quad a+b+c=x+y+z$

Substituting in the given inequality, we have to prove that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}>\frac{9}{x+y+z}$$

Now $\frac{x^{-1}+y^{-1}+z^{-1}}{3}>\left(\frac{x+y+z}{3}\right)^{-1}$ or $>\frac{3}{x+y+z}$

$$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}>\frac{9}{x+y+z}$$

(f) $\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}>\frac{9}{a+b+c}$

(Agra 1941, Annamalai 1949)

We know

$$\frac{(a+b)^{-1}+(b+c)^{-1}+(c+a)^{-1}}{3}>\left(\frac{a+b+b+c+c+a}{3}\right)^{-1}$$

or $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}>3\cdot\frac{3}{2(a+b+c)}$ etc.

(g) $\frac{3}{b+c+d}+\frac{3}{c+d+a}+\frac{3}{d+a+b}+\frac{3}{a+b+c}$

$$>\frac{16}{a+b+c+d}$$

Proceed as in part (f),

(h) $a^4+b^4+c^4>abc(a+b+c)$

$\frac{a^4+b^4+a^4}{3}>\left(\frac{a+b+c}{3}\right)^4$ or $>\left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3$

or $>\frac{a+b+c}{3}\{(abc)^{1/3}\}^3$ $\because$ A.M.>G.M.

or $>\left(\frac{a+b+c}{3}\right)abc$

$\therefore$ $a^4+b^4+c^4>abc(a+b+c)$

(i) $a^5+b^5+c^5+d^5>abcd(a+b+c+d)$

(Lucknow 40, Annamalai 47)

Proceed exactly as above.

*(j) $a^5+b^5+c^5>abc(ab+bc+ca)$

$\frac{a^5+b^5+c^5}{3}>\left(\frac{a+b+c}{3}\right)^5$ or $\left(\frac{a+b+c}{3}\right)^3\left(\frac{a+b+c}{3}\right)^2$

or $>[(abc)^{1/3}]^3\left(\frac{a^2+b^2+c^2+2ab+2bc+2ca}{9}\right)$

But we know that $a^2+b^2+c^2>ab+bc+ca$ by Ex. 1 p. 97.

$$\therefore \quad \frac{a^6+b^6+c^6}{3}>abc\frac{3(ab+bc+ca)}{9} \text{ etc.}$$

*(k) $\dfrac{a^8+b^8+c^8}{a^3b^3c^3}>\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

(Utkal 45, Agra 52)

or $a^8+b^8+c^8>a^2b^2c^2(bc+ca+ab)$

Now $$\frac{a^8+b^8+c^8}{3}>\left(\frac{a+b+c}{3}\right)^8 \text{ or } >\left(\frac{a+b+c}{3}\right)^6\left(\frac{a+b+c}{3}\right)^2$$

or $$>[(abc)^{1/3}]^6\left[\frac{a^2+b^2+c^2+2ab+2bc+2ca}{9}\right]$$

$\because$ A. M.>G. M.

But by Ex. 1 of page 97, $a^2+b^2+c^2>ab+bc+ca$

$$\therefore \quad \frac{a^8+b^8+c^8}{3}>a^2b^2c^2\,\frac{3(ab+bc+ca)}{9}$$

$$\therefore \quad a^8+b^8+c^8>a^2b^2c^2(ab+bc+ca)$$

or $$\frac{a^8+b^8+c^8}{a^3b^3c^3}>\frac{ab+bc+ca}{abc} \text{ or } >\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

*(l) $\dfrac{b^2+c^2}{b+c}+\dfrac{c^2+a^2}{c+a}+\dfrac{a^2+b^2}{a+b}>a+b+c$

(Nagpur 48)

$\dfrac{b^2+c^2}{2}>\left[\dfrac{b+c}{2}\right]^2$ $\because$ A.M. of mth powers > (A.M.)^m

$\therefore$ $\dfrac{b^2+c^2}{b+c}>\dfrac{b+c}{2}$. Similarly write other inequalities and add.

*(m) If $a_1^2+a_2^2+\dots\dots+a_n^2=A$, then
$nA>(a_1+a_2+\dots\dots a_n)^2>A$

(Allahabad 49, Agra 54 Supp)

$$\frac{a_1^2+a_2^2+\dots a_n^2}{n}>\left[\frac{a_1+a_2+\dots a_n}{n}\right]^2$$

or $$n(a_1^2+a_2^2+\dots a_n^2)>(a_1+a_2+\dots a_n)^2$$

or $$nA>(a_1+a_2+\dots a_n)^2$$

which is the 1st part of the inequality.

Again

$$(a_1+a_2+\ldots\ldots a_n)^2=a_1^2+a_2^2+\ldots\ldots a_n^2+2\Sigma\, a_1a_2$$

$$\therefore\quad (a_1+a_2+\ldots\ldots a_n)^2>a_1^2+a_2^2+\ldots\ldots a_n^2$$

because we have neglected all terms of the form $2a_1a_2$, $2a_1a_3$ and etc.

$$\therefore\quad (a_1+a_2+\ldots\ldots a_n)^2>A$$

which proves the 2nd part of the inequality.

***4. (a) Prove that**

$$(a_1^2+a_2^2+\ldots\ldots c_n^2)(b_1^2+b_2^2+\ldots\ldots b_n^2)$$
$$\geqslant(a_1b_1+a_2b_2+\ldots\ldots+a_nb_n)^2. \quad \text{(Pb. 42)}$$

We know that $(a_1x+b_1)^2$ is +ive i.e. $\geqslant 0$.

[being a perfect square]

Similarly $(a_2x+b_2)^2\geqslant 0$

$$(a_nx+b_n)^2\geqslant 0.$$

Adding, we get

$$(a_1^2+a_2^2+\ldots\ldots a_n^2)x^2+2x(a_1b_1+a_2b_2+\ldots\ldots+a_nb_n)$$
$$+(b_1^2+b_2^2+\ldots\ldots+b_n^2)\geqslant 0\ldots\ldots(1)$$

L. H. S. is of the form Ax^2+Bx+C which can be written as $\frac{1}{A}[A^2x^2+ABx+AC]$

or $\frac{1}{A}\left\{\left(Ax+\frac{B}{2}\right)^2+\frac{4AC-B^2}{4}\right\}$ which is positive only when $4AC-B^2$ is +ive i.e. $4AC\geqslant B^2$.........(2)

$\therefore$ L. H. S. is +ive if

$$4(a_1^2+a_2^2+\ldots\ldots a_n^2(b_1^2+b_2^2+\ldots\ldots b_n^2)$$
$$\geqslant 4(a_1b_1+a_2b_2+\ldots\ldots a_nb_n)^2 \text{ etc.}$$

4. (b) Prove that

$$(a^2+b^2+c^2)(x^2+y^2+z^2)>(ax+by+cz)^2$$

***5. If a, b, c are in H. P., prove that**

$$\frac{a+b}{2a-b}+\frac{c+b}{2c-b}>4.$$

$\because$ a, b, c are in H. P. $\therefore$ $b=\frac{2ac}{a+c}$

$$\therefore \frac{a+b}{2a-b}=\frac{a+\frac{2ac}{a+c}}{2a-\frac{2ac}{a+c}}=\frac{a^2+3ac}{2a^2}=\frac{1}{2}+\frac{3}{2}\frac{c}{a}$$

Similarly $\frac{c+b}{2c-b}=\frac{1}{2}+\frac{3}{2}\cdot\frac{a}{c}$

Hence we have to prove

$$\frac{1}{2}+\frac{3}{2}\left[\frac{c}{a}+\frac{a}{c}\right]+\frac{1}{2}>4$$

or $\frac{3}{2}\left[\frac{c}{a}+\frac{a}{c}\right]>3$ or $\frac{c^2+a^2}{2ac}>1$

or $\frac{c^2+a^2}{2}>ac$ which is true $\because$ A.M.>G.M.

***6.** If $S=a_1+a_2+\ldots\ldots a_n$, then

$$\frac{S}{S-a_1}+\frac{S}{S-a_2}+\ldots\ldots+\frac{S}{S-a_n}>\frac{n^2}{n-1}$$

(Agra 28, 42, 53, Ben. 37. Andhra 39, Alld. 41, Luck. 45)

Now A. M.>G. M.

$$\therefore \frac{S}{S-a_1}+\frac{S}{S-a_2}+\ldots\ldots+\frac{S}{S-a_n}$$

$$>n\left(\frac{S}{S-a_1}\cdot\frac{S}{S-a_2}\ldots\ldots\frac{S}{S-a_n}\right)^{1/n}$$

or $$>\frac{nS}{[(S-a_1)(S-a_2)\ldots\ldots(S-a_n)]^{1/n}}\ldots\ldots(1)$$

Now

$$\frac{(S-a_1)+(S-a_2)+\ldots\ldots(S-a_n)}{n}$$

$$>[(S-a_1)(S-a_2)\ldots(S-a_n)]^{1/n}$$

or $$\frac{n.S-(a_1+a_2+\ldots a_n)}{n}$$

$$>[(S-a_1)(S-a_2)\ldots(S-a_n)]^{1/n}$$

or $$\frac{(n-1)S}{n}>[(S-a_1)(S-a_2)\ldots(S-a_n)]^{1/n}$$

Taking reciprocal and hence changing the sign of inequality we get $$\frac{n}{(n-1)S}<\frac{1}{[(S-a_1)(S-a_2)\ldots(S-a_n)]^{1/n}}$$

or $$\frac{n}{(n-1)S}nS<\frac{nS}{[(S-a_1)(S-a_2)\ldots(S-a_n)]^{1/n}}$$

$$\therefore \frac{nS}{[(S-a_1)(S-a_2)\ldots(S-a_n)]^{1/n}}>\frac{n^2}{n-1}$$

Hence $$\frac{S}{S-a_1}+\frac{S}{S-a_2}+\ldots\ldots\frac{S}{S-a_n}>\frac{n^2}{n-1} \text{ from (1)}$$

***7. If n is a +ive integer and x<1, Show that**

$$\frac{1-x^{n+1}}{n+1}<\frac{1-x^n}{n}$$

(Agra 49, Allahabad 39)

We have to prove that

$$\frac{1-x^{n+1}}{1-x^n}<\frac{n+1}{n}$$

But $$\frac{1-x^{n+1}}{1-x^n}=\frac{1-x^n+x^n-x^{n+1}}{1-x^n}=1+\frac{x^n(1-x)}{1-x^n}$$

$$=1+\frac{x^n}{\frac{1-x^n}{1-x}}=1+\frac{x^n}{1+x+x^2+\ldots x^{n-1}}$$

$$=1+\frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\frac{1}{x^{n-2}}+\ldots\frac{1}{x}}\ldots\ldots(A)$$

Since $x<1$ $\therefore$ $\frac{1}{x}>1$

Similarly $\frac{1}{x^2}, \frac{1}{x^3} \ldots\ldots \frac{1}{x^n}$ are all >1

$\therefore$ $\frac{1}{x^n}+\frac{1}{x^{n-1}}+\frac{1}{x^{n-2}}+\ldots\ldots\frac{1}{x}>n$

Taking reciprocal and changing the sign of inequality, we get

$$\frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\frac{1}{x^{n-2}}+\ldots\ldots\frac{1}{x}}<\frac{1}{n}$$

or $1+\frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\frac{1}{x^{n-2}}+\ldots\ldots\frac{1}{x}}<1+\frac{1}{n}$

$\therefore$ by A,

$$\frac{1-x^{n+1}}{1-x^n}<\frac{n+1}{n}$$

or $$\frac{1-x^{n+1}}{n+1}<\frac{1-x^n}{n}$$

***8. If $a>b$ and n is a +ive integer. then prove**

$$(a^n-b^n)>n(ab)^{\frac{n-1}{2}}(a-b)$$

(Agra 32, 41, 50, Luck. 42)

Since $a>b$ $\therefore$ $(a-b)$ is +ive and hence we can divide the inequality by $(a-b)$ and we have now to prove that

$$\frac{a^n-b^n}{a-b}>n\ ab)^{\frac{n-1}{2}}$$

By actual division $\frac{a^n-b^n}{a-b}$

$=a^{n-1}+ba^{n-2}+b^2a^{n-3}+\ldots+a^2b^{n-3}+ab^{n-2}+b^{n-1}\ldots\ldots$(A)

Apply A. M. of the n numbers of A$>$their G. M.

$\therefore$ $A>n[a^{(n-1)+(n-2)\ldots\ldots3+2+1}b^{(n-1)+(n-2)\ldots3+2+1}]^{1/n}$

$$\text{or} \quad > n\left[(ab)^{\frac{n(n-1)}{2}}\right]^{\frac{1}{n}} \quad \text{or} \quad > n(ab)^{\frac{n-1}{2}}$$

***9,** $$\frac{\sqrt{(n+1)}}{2n+1} < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{(2n+1)}}$$

(Agra 47, Luck. 36, 47)

For second part squaring both sides, we have to prove that

$$\frac{1^2}{2^2}\,\frac{3^2}{4^2}\,\frac{5^2}{6^2}\cdots\cdots\frac{(2n-1)^2}{(2n)^2}(2n+1) < 1$$

$$\text{or} \quad \left(\frac{1.3}{2^2}\right)\left(\frac{3.5}{4^2}\right)\left(\frac{5.7}{6^2}\right)\cdots\cdots$$

$$\left[\frac{(2n-3)(2n-1)}{(2n-2)^2}\right]\left[\frac{(2n-1)(2n+1)}{(2n)^2}\right] < 1$$

which is clearly true as each bracket is less than 1 and hence their product is less than 1.

Again $n+(n+1) > 2\sqrt{[n(n+1)]}$ $\quad\because$ A. M.>G. M.

$$\text{or} \quad \frac{2n+1}{2n} > \sqrt{\left(\frac{n+1}{n}\right)} \quad \text{or} \quad \sqrt{\left(\frac{n+1}{n}\right)} < \frac{2n+1}{2n}$$

Put $n=1, 2, 3 \ldots (n-1), n$ and multiply the inequalities.

***10.** $$\frac{3.7.11\ldots\ldots\ldots(4n-1)}{5.9.13\ldots\ldots\ldots(4n+1)} < \sqrt{\left(\frac{3}{4n+3}\right)}$$

(Patna 37, Rajputana 50)

Proceed exactly as in Q. 9.

11. **If a, b, c are +ive and the sum of any two is greater than the third, then prove that**

$$\left(1+\frac{b-c}{a}\right)^a\left(1+\frac{c-a}{b}\right)^b\left(1+\frac{a-b}{c}\right)^c < 1\ldots\ldots\text{(A)}$$

$1+\frac{b-c}{a} = \frac{a+b-c}{a}$. Now $a+b>c$ given and a is +ive

$\therefore \frac{a+b-c}{a}$ is +ive

Now consider a factors each equal to $1+\frac{b-c}{a}$ and b

factors each equal to $1+\frac{c-a}{b}$ and c factors each equal to $1+\frac{a-b}{c}$, then because A. M.>G. M., we have

$$\frac{\left\{\left(1+\frac{b-c}{a}\right)+\left(1+\frac{b-c}{a}\right)+\ldots a \text{ times}\right\}\begin{array}{l}+(\ldots\ldots b \text{ times}).\\+(\ldots\ldots c \text{ times})\end{array}}{a+b+c}$$

$$>\left[\left\{\left(1+\frac{b-c}{a}\right)\ldots a \text{ factors}\right\}\left\{\left(1+\frac{c-a}{b}\right)\ldots b \text{ factors}\right\}\right.$$

$$\left.\left\{\left(1+\frac{a-b}{c}\right)\ldots c \text{ factors}\right\}\right]^{\frac{1}{a+b+c}}$$

$$\text{or}\quad\left\{\frac{a\left(\frac{a+b-c}{a}\right)+b\left(\frac{b+c-a}{b}\right)+c\left(\frac{c+a-b}{b}\right)}{a+b+c}\right\}^{a+b+c}$$

$$>\left(1+\frac{b-c}{a}\right)^{a}\left(1+\frac{c-a}{b}\right)^{b}\left(1+\frac{a-b}{c}\right)^{c}$$

$$\text{or}\quad\left(\frac{a+b+c}{a+b+c}\right)^{a+b+c}>\left(1+\frac{b-c}{a}\right)^{a}\left(1+\frac{c-a}{b}\right)^{b}\left(1+\frac{a-b}{c}\right)^{c}$$

$$\therefore\quad\left(1+\frac{b-c}{a}\right)^{a}\left(1+\frac{c-a}{b}\right)^{b}\left(1+\frac{a-b}{c}\right)^{c}<1$$

12. If there are n +ive numbers a_1, a_2,......a_n and if the square root of all their products taken together be found, prove that

$$\sqrt{(a_1a_2)}+\sqrt{(a_1a_3)}+\ldots\ldots<\frac{n-1}{2}(a_1+a_2+\ldots\ldots a_n)$$

and hence prove that the arithmetic mean of the square roots of the products taken two together is less than the arithmetic mean of given quantities.

Q. 11, 12 may be left on 1st reading.

Out of n letters the combination of 2 at a time will be ${}^nC_2$ i. e. $\frac{n(n-1)}{2}$

Now $\sqrt{(a_1a_2)}<\frac{a_1+a_2}{2}$ $\because$ G. M.<A. M.

$\sqrt{(a_1a_3)}<\frac{a_1+a_3}{2}$

... ... Adding, we get

$\sqrt{(a_1a_2)}+\sqrt{(a_1a_3)}+\ldots\ldots<\frac{1}{2}(n-1)(a_1+a_2+\ldots a_n)$

$\because$ In all these inequalities in R. H. S. $a_1, a_2, \ldots\ldots a_n$ all occur $(n-1)$ times each.

We have proved that

$$\left(\sqrt{(a_1a_2)}+\sqrt{(a_1a_3)}+\ldots\ldots\frac{n(n-1)}{2}\text{ terms}\right)$$
$$<\frac{n-1}{2}(a_1+a_2+\ldots a_n)$$

$$\therefore \frac{\sqrt{(a_1a_2)}+\sqrt{(a_1a_3)}+\ldots\frac{n(n-1)}{2}\text{ terms}}{\frac{n(n-1)}{2}}<\frac{a_1+a_2+\ldots a_n}{n}$$

or A. M. of square root of products taken two together out of n quantities is less than A.M. of the n quantities itself.

***13. Show that**

1943 $2(a^2+b^2+c^2)(a+b+c)>a^3+b^3+c^3+15abc$

$2(a^2+b^2+c^2)(a+b+c)>a^3+b^3+c^3-3abc+18abc$

(Note)

Now $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Transposing on the other side and taking $a+b+c$ common we have to prove that

$(a+b+c)(2a^2+2b^2+2c^2-a^2-b^2-c^2+ab+bc+ca)>18abc$

or $(a+b+c)(a^2+b^2+c^2+ab+bc+ca)>18abc$

Now

$(a+b+c)(a^2+b^2+c^4+ab+bc+ca)>3(abc)^{1/3}.6(a^4b^4c^4)^{1/6}$

$\because$ A. M.>G. M.

or $\quad > 18(abc)^{1/3}.(abc)^{2/3}$

or $\quad > 18\,abc$ Hence proved

***14 (a) Prove that**

$$(a+b+c+d)(a^3+b^3+c^3+d^3) > (a^2+b^2+c^2+d^2)^2$$

(Agra 44)

or we have to prove that

$$(a+b+c+d)(a^3+b^3+c^3+d^3)-(a^2+b^2+c^2+d^2)^2 > 0$$

i. e. +ive

Simplifying a^4, b^4, c^4, d^4 all cancel.

$(a^3b+b^3a-2a^2b^2)+$five similar expressions > 0

or $\quad ab(a^2-2ab+b^2)+$five similar expressions > 0

or $\quad ab(a-b)^2+\ldots\ldots > 0$

but $ab(a-b)^2$ is a +ive quantity, because a, b, c, d are all +ive. Similarly all other five expressions will be +ive which prove the theorem.

14 (b) Prove that $(10x^2+5y^2+13z^2) > 2(xy+4yz+9zx)$

(Pb. 48)

we have to prove that $10x^2+5y^2+13z^2-2xy-8yz-18xz > 0$

or $\quad (x^2+y^2-2xy)+4(y^2+z^2-2yz)+9(x^2+z^2-2xz) > 0$

which is true since each factor is a perfect square

***15. Prove that** **(Allahabad 36)**

$$\frac{a^2+b^2+c^2}{ab+bc+ca} > 1 \text{ and } < 2$$

Like Sum 1954

Now we know by Ex. Page 97.

$$a^2+b^2+c^2 > ab+bc+ca$$

$$\therefore \frac{a^2+b^2+c^2}{ab+bc+ca} > 1\ldots\ldots(A)$$

Again by Trigonometry we know that

$$b^2+c^2-a^2=2bc\cos A < 2bc \quad \therefore \quad \cos A < 1$$

Similarly $\quad c^2+a^2-b^2 < 2ca$

$$a^2+b^2-c^2 < 2ab$$

Adding we get $a^2+b^2+c^2 < 2(ab+bc+ca)$

$$\therefore \quad \frac{a^2+b^2+c^2}{ab+bc+ca} < 2 \ldots\ldots (B)$$

Hence proved,

***16. Prove that**

$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b) > 0$ i.e. +ive and $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) > 0$ i. e. +ive

Since the letters a, b, c in the above expression occur symetrically, we may suppose that $a > b > c$

$\therefore \quad c^2(c-a)(c-b)$ is +ive.........(A)

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)$$
$$=(a-b)\{a^3-ca^2-b^3+cb^2\}$$
$$=(a-b)\{(a^3-b^3)-c(a^2-b^2)\}$$
$$(a-b)^2\{a^2+b^2+ab-ca-cb\} \ldots\ldots\ldots (B)$$

Now $(a-b)^2$ being the square is +ive

ab is also +ive

$a^2-ac=a(a-c)$ is also +ive $\because \quad a > b > c$

$b^2-cb=b(b-c)$ is also +ive $\because \quad a > b > c$

Adding we get B is +ive

$\therefore$ A+B is also +ive

i. e. given expression is +ive i. e. > 0

The second expression may be similarly proved to be +ive.

17. Prove that $\dfrac{1}{m}\log(1+a^m) < \dfrac{1}{n}\log(1+a^n)$ if $m>n$

Case I: $a < 1$

then because $m > n \quad \therefore \quad a^m < a^n$

or $\quad 1+a^m < 1+a^n$

Taking log of both sides

$$\log(1+a^m) < \log(1+a^n) \ldots\ldots\ldots (A)$$

$$m > n \quad \therefore \quad \frac{1}{m} < \frac{1}{n} \ldots\ldots\ldots\ldots (B)$$

Q. 16, 17, 18, 19, 20, 21, 22 may be left on 1st reading.

Multiply the respective sides of A and B

we get $\frac{1}{m}\log(1+a^m)<\frac{1}{n}\log(1+a^n)$

Case II. $a>1$ then $\frac{1}{a}<1$

Put $\frac{1}{a}=A<1$ then by Case I

$$\frac{1}{m}\log(1+A^m)<\frac{1}{n}\log(1+A^n)$$

Putting $A=\frac{1}{a}$ we get

$$\frac{1}{m}\log\left(1+\frac{1}{a^m}\right)<\frac{1}{n}\log\left(1+\frac{1}{a^n}\right)$$

or $\frac{1}{m}[\log(1+a^m)-\log a^m]<\frac{1}{n}[\log(1+a^n)-\log a^n]$

or $\frac{1}{m}\log(1+a^m)-\frac{1}{m}.m\log a<\frac{1}{n}\log(1+a^n)-\frac{1}{n}n.\log a$

or $\frac{1}{m}\log(1+a^m)<\frac{1}{n}\log(1+a^n)$

18. Show that **(Allahabad 1949)**

$$(x^m+y^m)^n<(x^n+y^n)^m \text{ if } m>n$$

or we have to prove

$$x^{mn}\left[1+\left(\frac{y}{x}\right)^m\right]^n<x^{mn}\left[1+\left(\frac{y}{x}\right)^n\right]^m$$

Taking log of both sides

or $n\log\left[1+\left(\frac{y}{x}\right)^m\right]<m\log\left[1+\left(\frac{y}{x}\right)^n\right]$

Putting $\frac{y}{x}=a$

$n\log(1+a^m)<m\log(1+a^n)$

Dividing by mn, we have to prove that

$$\frac{1}{m}\log(1+a^m)<\frac{1}{n}\log(1+a^n)$$

which we have proved in question No. 18

19. **If a, b, c denote the sides of a triangle show that**

(i) $a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)\geqslant 0$

p, q, r being any real numbers.

(ii) $a^2yz+b^2zx+c^2xy$ cannot be $+$ive if $x+y+z=0$

(i) Let $p>q>r$ then $(p-q)(p-r)$ is $+$ive and $(r-p)(r-q)$ is also $+$ive because both the factors are $-$ive and $(q-r)(q-p)$ is $-$ive because one factor is $+$ive and the other $-$ive,

Let $a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)=\text{A}$

The expression A has got Ist and the last term $+$ive the middle is $-$ive.

Now a, b, c being the sides of a triangle

$\therefore\ b<a+c \quad \therefore\ b^2<(a+c)^2 \quad \ldots\ldots \quad \text{B}$

Now we know that if we multiply an inequality by a $-$ive quantity, the sign of inequality is changed.

Multiply both sides of B by the $-$ive quantity $(q-r)(q-p)$ and hence changing the sign we get

$b^2(q-r)(q-p)>(a+c)^2(q-r)(q-p)\ldots\ldots\ldots \quad \text{C}$

Adding Ist and 3rd terms of A in both sides of C we find that

$$\text{A}>a^2(p-q)(p-r)+(a+c)^2(q-r)(q-p)+c^2(r-p)(r-q)$$

$$\text{or }>a^2(p-q)\{(p-r)-(q-r)\}+c^2(r-q)\{(r-p)-(q-p)\}$$
$$+2ac(q-r)(q-p)$$

$$\text{or }>a^2(p-q)^2+c^2(r-q)^2+2ac(p-q)(r-q)$$

$$\text{or }>\{a(p-q)+c(r-q)\}^2$$

or $>$ a $+$ive quantity *i, e.* >0

$\therefore$ A>0 Proved.

(ii) We have to prove that if $x+y+z=0$, then

$$a^2yz+b^2zx+c^2xy \text{ cannot be } +\text{ive}$$

Now $z=-(x+y)$ $\quad\therefore$ We have to prove that

$$-(a^2y+b^2x)(x+y)+c^2xy \text{ cannot be } +\text{ive}$$

$$-[a^2y^2+b^2x^2+(a^2+b^2)xy]-c^2xy \text{ cannot be } +\text{ive}$$

or changing the sign

$$a^2y^2+b^2x^2-2abxy+(a^2+2ab+b^2)xy-c^2xy \text{ cannot be } -\text{ive.}$$

or $(ay-bx)^2+[(a+b)^2-c^2]xy$ cannot be $-$ive.

Now $(ay-bx)^2$ being a perfect square cannot be negative ......(A)

In a triangle $a+b>c$ $\quad\therefore$ $(a+b)^2-c^2$ is $+$ive.

$\therefore$ $[(a+b)^3-s^2]xy$ cannot be $-$ive.........(B)

Adding A and B we find that

$$(ay-bx)^2+[(a+b)^2-c^2]xy \text{ cannot be } -\text{ive.}$$

20. If x is +ive, then show that

$$\log(1+x)<x \text{ and } >\frac{x}{1+x}$$

Now $\quad\log(1+x)$ will be less than x.

If $1+x<e^x$ or $<1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ which is quite clear.

Again $\log(1+x)$ will be greater than $\frac{x}{1+x}$

If $1+x>e^{\frac{x}{1+x}}$

or If $\frac{1}{1-\frac{x}{1+x}}>e^{\frac{x}{1+x}}$

or If $\left(1-\frac{x}{1+x}\right)^{-1}>e^{\frac{x}{1+x}}$

or If $$1+\frac{x}{1+x}+\frac{x^2}{(1+x)^2}+\ldots\ldots>1+\frac{x}{1+x}+\frac{x^2}{2!(1+x)^2}+\ldots\ldots$$

which is quite clear, because all the terms on the L. H. S. after the second term are greater than the corresponding terms on the R. H. S. Hence proved.

21. **Prove that**

$$(n!)^2<r!\,.\,(2n-r)! \text{ where } r<n.$$

or we have to prove that $\frac{r!\,(2n-r)!}{n!}>n!$

Now $$\frac{r!\,(2n-r)!}{n!}=\frac{1.2.3\ldots r(2n-r)(2n-r-1)\ldots(n+1)(n!)}{n!}$$

$$=(1.2.3\ldots\ldots r)\{(n+1)(n+2)\ldots(2n-r)\}$$

$$>(1.2.3\ldots\ldots r)\{(r+1)(r+2)\ldots\ldots n\}$$

$\because$ $n>r$ $\therefore$ $(n+1)>(r+1)$ etc.

and $n-r>0$ $\therefore$ $2n-r>n$

or $>1.2.3\ldots\ldots r(r+1)(r+2)\ldots\ldots n$ or $>n!$

$\therefore$ $\frac{r!\,(2n-r)!}{n!}>n!$

$r!\,(2n-r)!>(n!)^2$ $\therefore$ $(n!)^2<r!\,.\,(2n-r)!$

22. **Show that**

$$1!\;3!\;5!\ldots\ldots(2n-1)!>(n!)^n$$

(Agra 45, Benares 48, Allahabad 50)

We have proved in Q. 20 that

$$r!\,(2n-r)!>(n!)^2 \quad n \text{ being } > r$$

Give r successively the values 1, 3, 5......$(2n-1)$

$1!\,(2n-1)!>(n!)^2$

$3!\,(2n-3)!>(n!)^2$

$5!\,(2n-5)!>(n!)^2$

... ...

$(2n-1)!\;1!>(n!)^2$

Multiplying the respective sides we get

$\{1!\,3!\,5!\,7!\ldots\ldots(2n-1)!\}^2>(n!)^{2n}$

Taking square root of both sides.

$1!\,3!\,5!\ldots\ldots(2n-1)!>(n!)^n$

23, Find the Max. value of $(8-x)^3(x+6)^4$ when it is given that x lies between -6 and 8

Since x lies between -6 and 8 both $8-x$ and $x+6$ are +ive

Hence proceeding exactly as in Ex. 1 of Page 108 we find that Max. value of $(8-x)^3(x+6)^4$ is $6^3 \cdot 8^4$.

(b) If $2x+5y=3$. find the Maximun value of x^3y^4
Proceeding exactly as in Ex. 2 of Page 109 we get

Max. value of x^3y^4 is $\left(\frac{3^2}{2.7}\right)^3\left(\frac{3.4}{5.7}\right)^4$

$$=\frac{3^{10}.2^5}{5^4.7^7}$$

(c) Find the Max. value of $(7-x)^4(2+x)^5$ when x lies between 7 and -2,

Because x lies between 7 and -2, both $7-x$ and $2+x$ are +ive, hence proceeding exactly as in Ex. 1 of Page 108 we find that the Max. value of $(7-x)^4(2+x)^5$ is $4^4.5^5$

(d) Find the Max. value of $x^3(4a-x)^5$ if x is +ive and less then $4a$.

Since x is +ive and less than $4a$ then both x and $4a-x$ are +ive. Hence proceeding as in part a and c we find that the Max. value of $x^3.(4a-x)^5$ becomes.

$$\left[\frac{3a}{2}\right]^3\left[\frac{5a}{2}\right]^5=3^3.5^5\ \frac{a^8}{2^8}$$

(e) Find the Max. value of $x^{1/2}\,[1-x]^{1/3}$ when x is a

proper fraction.

Because x is a proper fraction.

$\therefore$ both x and $1-x$ are +ive

Now $x^{1/2}[1-x]^{1/3}$ is Max. when

$x^3(1-x)^2$ is Max. [on taking sixth power]

and $x^3(1-x)^2$ is Max. when

$$\left[\frac{x}{3}\right]^3\left[\frac{1-x}{2}\right]^2 \text{ is Max.......A}$$

But A is the product of $3+2=5$ factors whose sum is

$$3.\frac{x}{3}+2.\frac{1-x}{2}=1=a \text{ constant}$$

and hence by art 6, the product will be Max. when

$$\frac{x}{3}=\frac{1-x}{2}=\tfrac{1}{5} \text{ or when } x=\tfrac{3}{5}$$

Max. value of $x^{1/2}(1-x)^{1/3}$ is $\sqrt{\tfrac{3}{5}}\cdot(\tfrac{2}{5})^{1/3}$

i.e. $\sqrt{\tfrac{3}{5}}\cdot\sqrt[3]{\tfrac{2}{5}}$

***(f) Find the minimum value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ for +ive values of x, y and z which satisfy the condition $x+y+z=k$** **(Agra 54)**

Now by art 11 (3a) i.e. if $X+Y+Z\ldots\ldots=\text{const}=S$ then the least value of $X^m+Y^m+Z^m+\ldots\ldots$ occurs when $X=Y=Z\ldots\ldots=\frac{S}{n}$ and the least value becomes

$$n\left(\frac{S}{n}\right)^m$$

Here $x^{-1}+y^{-1}+z^{-1}$ will have least value when

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=\frac{x+y+z}{3}=\frac{k}{3}$$

and the value will be $\left(\frac{k}{3}\right)^{-1}+\left(\frac{k}{3}\right)^{-1}+\left(\frac{k}{3}\right)^{-1}$

$$=3\left(\frac{3}{k}\right)=\frac{9}{k}.$$

***(g) Find the Minimum value of bcx+cay+abz when** $xyz=abc$

Put $bcx=X$, $cay=Y$ and $abz=Z$

then we are to find the Min. value of $X+Y+Z$ when

$$\frac{X}{bc}\cdot\frac{Y}{ca}\cdot\frac{Z}{ab}=abc\ [\because\ xyz=abc]$$

i. e. $XYZ=a^3b^3c^3=$const.

Now by article 11 (2) if $XYZ......=$Const. $=P$ then the value of $X+Y+Z+\ ....$ is least when $X=Y=Z=P^{1/n}$

Now $X+Y+Z$ will be least when

$$X=Y=Z=(XYZ)^{1/3}=(a^3b^3c^3)^{1/3}=abc$$

and hence least value is $abc+abc+abc=3abc$

***(h) Prove that the equilateral triangle has max. area for given perimeter and min. perimeter for given area.** **(Nagpur 38)**

Let a, b, c be the sides then $2s=a+b+c$

Area $A=\sqrt{[s(s-a)(s-b)(s-c)]}$

Perimeter $P=a+b+c=2s$

Put $s-a=x$, $s-b=y$, $s-c=z$

then adding $3s-(a+b+c)=x+y+z$ i.e. $s=x+y+z$

1. Perimeter constant

$$A=\sqrt{[(x+y+z)xyz]}=\sqrt{(sxyz)}$$
$$s=(x+y+z)=\text{Constant}$$

Now if Perimeter is constant then $x+y+z$ is constant then A will be Max. when A^2 or when $sxyz$ is Max. but s being constant, we should find the max. value of xyz which will be by Art. 11 (1) when $x=y=z$ or when

$s-a=s-b=s-c$ or $a=b=c$ i.e. $\triangle$ is equilateral

II. Area Constant

$$A^2=(x+y+z)xyz=x^2yz+y^2zx+z^2xy$$
$$=X+Y+Z \text{ where } X=x^2yz, \text{ etc.)}$$

$$P=2s=2(x+y+z)=2\left(\frac{X}{xyz}+\frac{Y}{xyz}+\frac{Z}{xyz}\right)$$

$$=\frac{2A^2}{xyz}=\frac{2A^2}{(XYZ)^{1/4}} \qquad \because \quad XYZ=x^4y^4z^4$$

Now area is constant i. e. $X+Y+Z$ is constant and then perimeter will be minimum when $\dfrac{2A^2}{(XYZ)^{1/4}}$ is minimum or $(XYZ)^{1/4}$ is Max. or XYZ is Max. which by art 11 (1) will be Max. only when $X=Y=Z$

i. e. $x^2yz=y^2zx=z^2xy$ or $x=y=z$ or $a=b=c$ i. e. $\triangle$ is equilateral

(i) Find the Max. Value of xyz when

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Let $\dfrac{x^2}{a^2}=X,\ \dfrac{y^2}{b^2}=Y,\ \dfrac{z^2}{c^2}=Z$

$\therefore$ we are given that $X+Y+Z=1$ a constant and we are to find Max. value of xyz i. e. $abc\sqrt{(XYZ)}$.

Now $abc\sqrt{(XYZ)}$ is Max. when XYZ is Max. and XYZ is Max. when $X=Y=Z=\frac{1}{3}$ because $X+Y+Z=1$ constant given.

$$X=\frac{x^2}{a^2}=\frac{1}{3}\quad\therefore\quad x=\frac{a}{\sqrt{3}}$$

Similarly $y=\frac{b}{\sqrt{3}}$

and $z=\frac{c}{\sqrt{3}}$

$\therefore$ Max. value of xyz is $\frac{abc}{3\sqrt{3}}$

(j) If a, b, c are +ive rationals and x, y, z are +ive quantities, such that $x+y+z$ is constant, show that

$x^a y^b z^c$ is maximum when $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ and its value

$$a^a b^b c^c\left(\frac{x+y+z}{a+b+c}\right)^{a+b+c}$$

(Agra 29)

Now $x^a . y^b . z^c$ is Max. when $\left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c$ is Max........(A)

Now A is product of $a+b+c$ factors whose sum is

$$a\left(\frac{x}{a}\right)+b\left(\frac{y}{b}\right)+c\left(\frac{z}{c}\right)=x+y+z=\text{const.}$$

(given)

and by article No. 3 we know that the greatest value of a product, sum of whose factors is constant, is when all the factors are equal. $\therefore$ for Max. value of A, and hence of $x^a . y^b . z^c$, we must have

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{x+y+z}{a+b+c}$$

Thus the max. value of $x^a y^b z^c$ is

$$a^a b^b c^c\left(\frac{x+y+z}{a+b+c}\right)^{a+b+c}$$

***(k) Find the greatest value of**

$$(a-x\ (b-y)(c-z)(ax+by+cz)$$

where a, b, c **are known +ive quantities and** $a-x$, $b-y$, $c-z$ **are also +ive.** **(Agra 38)**

Now given expression is greatest when the numerator of

$$\frac{(a^2-ax)(b^2-by)(c^2-cz)(ax+by+cz)}{abc} \text{ is greatest.}$$

The numerator being the product of four factors whose sum is $a^2+b^2+c^2$ i e. constant because a, b, c are known and hence will be maximum when all the factors are equal. (Art. 5)

i.e. $$\frac{a^2-ax}{1}=\frac{b^2-by}{1}=\frac{c^2-cz}{1}=\frac{ax+by+cz}{1}=\frac{a^2+b^2+c^2}{4}$$

Hence the Max. value of the given expression is

$$\frac{1}{abc}\left[\frac{a^2+b^2+c^2}{4}\right]^4.$$

CHAPTER IV

CONVERGENCY AND DIVERGENCY

§1. An expression in which the successive terms are formed by some definite law is called a series. If the number of terms is finite then the series is called Finite series and if the number of terms is unlimited then it is called an Infinite Series. We shall in this chapter deal with infinite series and as and when we say a series in this chapter we shall mean an **Infinite series** which will be denoted as

$u_1+u_2+u_3+\dots\dots u_n+\dots\dots$ which is written in short from as $\sum\limits_{n=1}^{\infty} u_n$ or $\sum a_n$

The sum of the first n terms of the above series will be denoted by S_n. For example the following series are infinite series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots\frac{1}{n^2}+\dots\dots$$

$$1+x+x^2+x^3+\dots\dots x^{n-1}+\dots\dots$$

§2. Definition of Convergency and Divergency.

Convergency :

A series is said to be convergent if the sum of its first n terms is such that it is always less than or does not exceed some finite quantity, however great n may be. The above statement is equivalent to saying that a series will be convergent if the sum of its first n terms tends to a finite limit as n tends to infinity and is expressed as

If $\underset{n\to\infty}{\text{Lt}}\ S_n=$ finite the series is Convergent.

When a series is convergent and hence $\underset{n\to\infty}{\mathrm{Lt}} S_n =$ finite quantity say (s) then s is called the sum of the series to infinity.

Divergent :

A series is said to be divergent if the sum of its first n terms is such that it can always be made to exceed any finite quantity whatsoever you can think of, by taking n sufficiently great. The above statement is equivalent to saying that a series will be divergent if the sum of its first n terms tends to either plus infinity or minus infinity as n tends to infinity and is expressed as if $\underset{n\to\infty}{\mathrm{Lt}} S_n = +\infty$ or $-\infty$ the series is divergent.

Oscillating :

A series the sum of whose first n terms is such that it neither tends to a finite limit nor to plus, minus infinity as n tends to infinity is said to be Oscillating series.

Ex. Consider a G. P.

$$1+x+x^2+x^3+\dots\dots x^{n-1}+\dots\dots$$

$$S_n=\frac{1-x^n}{1-x} \text{ when } x<1$$

and $$\frac{x^n-1}{x-1} \text{ when } x>1$$

Now suppose x is numerically less than unity.

Say $=\frac{1}{2}$, then $S_n=\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}=2[1-(\frac{1}{2})^n]$

$1-(\frac{1}{2})^n$ will always be less than unity however great n may be and hence $2[1-(\frac{1}{2})^n]$ will always be less than a finite quantity 2. Hence S_n is such that it does not exceed a finite quantity 2 however great n may be and hence by our definition the series is convergent.

Again $\underset{n\to\infty}{\text{Lt}} S_n = \underset{n\to\infty}{\text{Lt}} 2[1-(\frac{1}{2})^n]=2$ i. e. finite quantity and hence our series is convergent.

Therefore remember that an infinte G P. is always convergent when its common ratio is numerically less than unity.

Again suppose x is numerically greater than unity say $=2$. then $S_n = \frac{2^n-1}{2-1} = 2^n - 1$

Hence S_n is such that it can be made to exceed any finite quantity by giving larger and larger values to n. Suppose $n=4$ then $S_n=15$. But S_n can be made to exceed 15 by giving n the value 7 and so on. Hence by our definition the series is divergent.

Again $\underset{n\to\infty}{\text{Lt}} S_n = \underset{n\to\infty}{\text{Lt}} \frac{(2^n-1)}{2-1} = \infty$

and hence our series is divergent.

Even if x =1 then $S_n = n$ and $\therefore$ $\underset{n\to\infty}{\text{Lt}} S_n = \underset{n\to\infty}{\text{Lt}} n = \infty$

Therefore remember that an infinite G. P. is always divergent when its common ratio is numerically greater than unity or unity itself

If $x=-1$ the series becomes

$1-1+1-1+1-1+\ldots\ldots$ then

$S_n=1$ or 0 according as n is odd or even

Thus S_n fluctuates between the values 0 and 1.

i. e. it neither tends to a finite limit nor to plus minus infinity and hence by our definition the series is oscillating.

Find by definition whether the following series are convergent or divergent.

Example 1. $1^3+2^3+3^3+\ldots\ldots\ldots n^3+\ldots\ldots\infty$

$$S_n=\left\{\frac{n(n+1)}{2}\right\}^2$$

$\underset{n\to\infty}{\mathrm{Lt}}\ S_n=\infty$ Hence the series is divergent.

Ex. 2. $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\ldots\ldots\infty$

It is a G. P. whose common ratio is $\frac{2}{3}$

$$S_n=\frac{1\{1-(\frac{2}{3})^n\}}{1-\frac{2}{3}}=3\{1-(\tfrac{2}{3})^n\}$$

$$\underset{n\to\infty}{\mathrm{Lt}}\ S_n=\underset{n\to\infty}{\mathrm{Lt}}\ 3\{1-(\tfrac{2}{3})^n\}=3\{1-0\}$$

$=3$ a finite quantity

$\therefore$ Series is convergent

Ex. 3. $\frac{1}{a(a+2)}+\frac{1}{(a+2)(a+4)}+\frac{1}{(a+4)(a+6)}+\ldots\ldots\ldots$

$$u_1=\frac{1}{a(a+2)}=\frac{1}{2}\left[\frac{1}{a}-\frac{1}{a+2}\right]$$

$$u_2=\frac{1}{(a+2)(a+4)}=\frac{1}{2}\left[\frac{1}{a+2}-\frac{1}{a+4}\right]$$

$$u_3=\frac{1}{(a+4)(a+6)}=\frac{1}{2}\left[\frac{1}{a+4}-\frac{1}{a+6}\right]$$

……………………………………………

$$u_n\qquad=\frac{1}{2}\left[\frac{1}{a+2(n-1)}-\frac{1}{a+2n}\right]$$

Adding and cancelling diagonally we get

$$S_n=\frac{1}{2}\left[\frac{1}{a}-\frac{1}{a+2n}\right]$$

$\underset{n\to\infty}{\mathrm{Lt}}\ S_n=\frac{1}{2}\left[\frac{1}{a}-0\right]=\frac{1}{2a}$ which is finite

Hence the series is convergent :—

§3. We have seen in the above example that in order to find out whether a series is convergent or divergent we have to take limit. So we must be fully conversant with the method of evaluating limits of functions of n when n tends to infinity.

Properties of limits.

If $\underset{n\to\infty}{\mathrm{Lt}}\, f(n)=a$ and $\underset{n\to\infty}{\mathrm{Lt}}\, \phi(n)=b$ then

1. $\underset{n\to\infty}{\mathrm{Lt}}\, \{f(n)+\phi(n)\}=a+b$
2. $\underset{n\to\infty}{\mathrm{Lt}}\, \{f(n)-\phi(n)\}=a-b$
3. $\underset{n\to\infty}{\mathrm{Lt}}\, \{kf(n)\}=ka$, where k is a constant
4. $\underset{n\to\infty}{\mathrm{Lt}}\, \{f(n)\phi(n)\}=ab$
5. $\underset{n\to\infty}{\mathrm{Lt}}\, \left\{\dfrac{f(n)}{\phi(n)}\right\}=\dfrac{a}{b}$
6. If $f(n)<\phi(n)$ then $\underset{n\to\infty}{\mathrm{Lt}}\, f(n)\leqq\underset{n\to\infty}{\mathrm{Lt}}\, \phi(n)$

Note the sign of equality.

i. e. suppose $f(n)=\dfrac{n+7}{3n}$ and $\phi(n)=\dfrac{n+11}{3n}$

Here $f(n)$ is less than $\phi(n)$

$$\underset{n\to\infty}{\mathrm{Lt}}\, f(n)=\underset{n\to\infty}{\mathrm{Lt}}\, \frac{n+7}{3n}=\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{1}{3}+\frac{7}{3n}\right]=\frac{1}{3}$$

$$\underset{n\to\infty}{\mathrm{Lt}}\, \phi(n)=\underset{n\to\infty}{\mathrm{Lt}}\, \frac{n+11}{3n}=\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{1}{3}+\frac{11}{3n}\right]=\frac{1}{3}$$

i. e. $\underset{n\to\infty}{\mathrm{Lt}}\, f(n)=\underset{n\to\infty}{\mathrm{Lt}}\, \phi(n)$

§4. Remember the following results.

4.1 If $x<1$ $\underset{n\to\infty}{\mathrm{Lt}}\, x^n=0$ i. e. $\mathrm{Lt}\,(\frac{1}{3})^n=0$

and if $x>1$ $\underset{n\to\infty}{\mathrm{Lt}}\, x^n=\infty$ i. e. $\mathrm{Lt}\,(3)^n=\infty$

4.2 If p is +ive then $\underset{n\to\infty}{\mathrm{tL}}\, n^p=\infty$

$$\text{and } \underset{n\to\infty}{\text{Lt}} \frac{1}{n^p}=0$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \text{ of } \sqrt{n},\ n^2,\ n^3 \text{ and so on}=\infty$$

$$\text{and } \underset{n\to\infty}{\text{Lt}} \frac{1}{n},\ \frac{1}{n^2},\ \frac{1}{\sqrt{n}}=0$$

4.3 $\underset{n\to\infty}{\text{Lt}} \dfrac{\log n}{n}=0$

Proof Put $\log n=k$ $\therefore$ $n=e^k$ and as $n\to\infty$, k also$\to\infty$

$$\underset{k\to\infty}{\text{Lt}} \frac{\log n}{n}=\underset{k\to\infty}{\text{Lt}} \frac{k}{e^k}=\underset{k\to\infty}{\text{Lt}} \frac{k}{1+k+\frac{k^2}{2\,!}+\frac{k^3}{3\,!}+\ldots\ldots}$$

$$=\underset{k\to\infty}{\text{Lt}} \frac{1}{\frac{1}{k}+1+\frac{k}{2\,!}+\frac{k^2}{3\,!}+\ldots\ldots}=\frac{1}{\infty}=0$$

4.4 $\underset{n\to\infty}{\text{Lt}} \left(1+\dfrac{1}{n}\right)^n=e$

The power n is exactly the reciprocal of the 2nd term in the bracket $\dfrac{1}{n}$

(a) $$\underset{n\to\infty}{\text{Lt}} \left(1-\frac{1}{n}\right)^n=\underset{n\to\infty}{\text{Lt}} \left[\left\{1+\left(\frac{-1}{n}\right)\right\}^{-n}\right]^{-1}$$

$$=e^{-1}=\frac{1}{e}$$

(b) $$\underset{n\to\infty}{\text{Lt}} \left(1+\frac{2}{n}\right)^n=\underset{n\to\infty}{\text{Lt}} \left\{\left(1+\frac{2}{n}\right)^{n/2}\right\}^2=e^2$$

$$\underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{2n}\right)^n=\underset{n\to\infty}{\text{Lt}} \left\{\left(1+\frac{1}{2n}\right)^{2n}\right\}^{1/2}=e^{1/2}$$

(d) $$\underset{n\to\infty}{\mathrm{Lt}}\left(1-\frac{3}{n}\right)^n=\underset{n\to\infty}{\mathrm{Lt}}\left\{\left(1-\frac{3}{n}\right)^{-n/3}\right\}^{-3}=e^{-3}$$

4.5. The most important case is of evaluating the limits of the function of the form $\dfrac{f(n)}{\phi(n)}$.

(a) If $f(n)$ and $\phi(n)$ are of the same degree say k, then $$\underset{n\to\infty}{\mathrm{Lt}}\frac{f(n)}{\phi(n)}=\frac{\text{coefficient of } n^k \text{ in } f(n)}{\text{coefficient of } n^k \text{ in } \phi(n)}$$

Ex. 1. $$\underset{n\to\infty}{\mathrm{Lt}}\frac{n^2+3n+4}{3n^2+4n+7}=\frac{1}{3}$$

because on dividing numerator and denominator by n^2, the

$$\text{above limit}=\underset{n\to\infty}{\mathrm{Lt}}\frac{1+\frac{3}{n}+\frac{4}{n^2}}{3+\frac{4}{n}+\frac{7}{n^2}}=\frac{1+0+0}{3+0+0}=\frac{1}{3}$$

Ex. 2. $$\underset{n\to\infty}{\mathrm{Lt}}\frac{(2n+3)(4n+5)(n+4)}{(5n+7)(6n+11)(3n+4)}$$

Here both numerator and denominator are of third degree and hence above limit

$$=\frac{\text{coefficient of } n^3 \text{ in the numerator}}{\text{coefficient of } n^3 \text{ in the denominator}}=\frac{2.4.1}{5.6.3}=\frac{4}{45}$$

(b) If the degree of numerator $f(n)$ is greater than that of the denominator $\phi(n)$, then

$$\underset{n\to\infty}{\mathrm{Lt}}\frac{f(n)}{\phi(n)}=\infty$$

Ex 3 $$\underset{n\to\infty}{\mathrm{Lt}}\frac{n^2+2n+7}{3n+4}=\infty$$

Dividing numerator and denominator by highest degree term of n i.e. n^1 the above limit.

$$=\underset{n\to\infty}{\text{Lt}}\frac{1+\frac{2}{n}+\frac{7}{n^2}}{\frac{3}{n}+\frac{4}{n^2}}=\frac{1+0+0}{0+0}=\frac{1}{0}=\infty$$

(c) If the degree of numerator $f(n)$ is less than that of the denominator $\phi(n)$, then

$$\underset{n\to\infty}{\text{Lt}}\frac{f(n)}{\phi(n)}=0$$

Ex. 4. $\underset{n\to\infty}{\text{Lt}}\dfrac{3n+4}{n^2+2n+4}$

Dividing the numerator and denominator by highest degree term of n i.e. n^2, the above limit

$$=\underset{n\to\infty}{\text{Lt}}\frac{\frac{3}{n}+\frac{4}{n^2}}{1+\frac{2}{n}+\frac{4}{n^2}}=\frac{0+0}{1+0+0}=0$$

4·6. If $x<1$, then $\underset{n\to\infty}{\text{Lt}}\, nx^n=0$.

Proof. Because $x<1$ $\therefore$ $\dfrac{1}{x}>1$

$\therefore$ $\left(\dfrac{1}{x}\right)^n$ tends to infinity as n tends to ∞

Put $\left(\dfrac{1}{x}\right)^n=t$. Then as $n\to\infty$, t also $\to\infty$

Taking log of both sides,

$$n\log\left(\frac{1}{x}\right)=\log t \quad \therefore \quad n=\frac{\log t}{\log\left(\frac{1}{x}\right)}$$

Hence $nx^n=\dfrac{n}{\left(\frac{1}{x}\right)^n}=\dfrac{\log t}{t\log\left(\frac{1}{x}\right)}$

$$\therefore \underset{n\to\infty}{\mathrm{Lt}}\, nx^n = \underset{t\to\infty}{\mathrm{Lt}} \frac{\log t}{t\log\left(\frac{1}{x}\right)}$$

But by 4·3 $\underset{t\to\infty}{\mathrm{Lt}} \frac{\log t}{t}=0$ and $\log\left(\frac{1}{x}\right)$ remains constant when n and t change

$$\therefore \underset{n\to\infty}{\mathrm{Lt}}\, nx^n=0$$

§5. Before proceeding to find out certain tests for the determination of convergency and divergency of series, we shall find the following theorems, to be of great help. (Read)

(a) $\underset{n\to\infty}{\mathrm{Lt}} (a+b+c+u_1+u_2+u_3\ldots u_n)$ where a, b, c are all finite quantities $=(a+b+c)+\underset{n\to\infty}{\mathrm{Lt}} S_n$ $=$finite or infinite according as $\underset{n\to\infty}{\mathrm{Lt}} S_n=$finite or infinite.

Hence the series $a+b+c+u_1+u_2+u_3\ldots\ldots\ldots\ldots\infty$ is convergent or divergent according as $u_1+u_2+u_3\ldots\ldots\infty$ is convergent or divergent.

(b) $\underset{n\to\infty}{\mathrm{Lt}} \quad (u_{m+1}+u_{m+2}+\ldots\ldots u_n)$

$$=\underset{n\to\infty}{\mathrm{Lt}} \quad (u_1+u_2+\ldots\ldots+u_{m+1}+u_{m+2}\ldots\ldots u_n)$$
$$-(u_1+u_2+\ldots\ldots u_m)$$

$=\underset{n\to\infty}{\mathrm{Lt}} S_n-(u_1+u_2+u_3\ldots\ldots u_m)=$finite and infinite according as $\underset{n\to\infty}{\mathrm{Lt}} S_n=$finite or infinite

because $u_1+u_2\ldots\ldots u_m$ being the sum of some finite number of terms and hence equal to some constant.

Hence the series $u_{m+1}+u_{m+2}+\ldots u_n+\ldots$ is convergent or divergent according as

$$u_1++u_2+\ldots u_m+u_{m+1}+u_{m+2}+\ldots u_n+\ldots$$

is convergent or divergent.

(e) $\underset{n\to\infty}{\text{Lt}}$ $Ku_1+Ku_2+\ldots Ku_n$ where K is any constant$=K.\underset{n\to\infty}{\text{Lt}}$ $S_n=$finite or infinite according as $\underset{n\to\infty}{\text{Lt}}$ S_n - finite or infinite.

Hence the series $Ku_1+Ku_2+\ldots Ku_n+$. where K is any constant is convergent or divergent according as $u_1+u_2\ldots\ldots u_n$ is convergent or divergent.

The above three results can be summed up as follows.

The nature of a series remains unaltered by the addition or removal of a finite number of terms or by the multiplication of each term by some constant.

Note :—By definition we can test the convergency and divergency of a series only by finding the sum of its first n terms and then taking its limit when n tends to infinity. But it may happen that we may not be able to find the sum of first n terms. Therefore we shall give below certain tests, with the help of which we shall be able to investigate the Convergency and Divergency of a given series.

§6. An infinite series in which the terms are alternately +ive and −ive is convergent, if each term is numerically less than the preceeding term and if the terms decrease indefinitely. (Agra 29, Pb. 32, 43)

i. e. if $\underset{n\to\infty}{\text{Lt}}$ $u_n=0$

Proof. Let the series be

$$u_1-u_2+u_3-u_4+u_5-u_6\ldots\ldots$$

where $u_1 > u_2 > u_3 > u_4 \ldots\ldots$ and $\underset{n \to \infty}{\mathrm{Lt}}\ u_n = 0$

Let us consider the sum of $2n$ terms

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots\ldots (u_{2n-1} - u_{2n}) \ldots\ldots \text{(A)}$$

It can also be put in the form

$$S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots\ldots - u_{2n} \ldots\ldots$$

$$= u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots\ldots + u_{2n}] \ldots\ldots \text{(B)}$$

Since $(u_1 - u_2)$, $(u_3 - u_4) \ldots\ldots$ and all such brackets are +ive by the given conditions and hence from (A) we find that S_{2n} must be +ive. With the support of this finding we conclude from (B) that S_{2n} is always less than u_1 a finite quantity, because the terms within the brackets in (B) cannot be greater than u_1 for if they are, then S_{2n} will be −ive which will be contrary to our finding from the form (A). Now S_{2n} being always less than u_1 $\therefore \underset{n \to \infty}{\mathrm{Lt}}\ S_{2n} =$ finite quantity

Again $S_{2n+1} = S_{2n} + u_{2n+1}$

$$\therefore \underset{n \to \infty}{\mathrm{Lt}}\ S_{2n+1} = \underset{n \to \infty}{\mathrm{Lt}}\ S_{2n} + \underset{n \to \infty}{\mathrm{Lt}}\ u_{2n+1}$$

But because $\underset{n \to \infty}{\mathrm{Lt}}\ u_n = 0$ $\therefore \underset{n \to \infty}{\mathrm{Lt}}\ u_{2n+1} = 0$

Hence we find that S_{2n+1} tends to the same finite limit as S_{2n}. i. e. $\underset{n \to \infty}{\mathrm{Lt}}\ S_n = a$ finite quantity whether n is even or odd and therefore by definition the series is convergent.

Note—The students should always certify whether $\underset{n \to \infty}{\mathrm{Lt}}\ u_n = 0$ or not.

Ex. 1. Test the convergency of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots\ldots$$

u_n is numerically equal to $\frac{1}{n}$

$$\therefore \underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{1}{n} = 0$$

The terms are alternately +ive and −ive and also each term is numerically less than the preceding term and hence the series is convergent.

Ex. 2. $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\ldots\ldots$ **(Pb. 41, Agra 44)**

The terms are alternately +ive and − ive and also each term is numerically less than the preceding term

But $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{n+1}{n} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right) = 1$

which is not zero. Therefore the series is not convergent.

We can put the series as

$$(1+1)-(1+\tfrac{1}{2})+(1+\tfrac{1}{3})-(1+\tfrac{1}{4})+\ldots\ldots$$

$$(1-1+1-1+\ldots\ldots)+(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots\ldots)$$

$$=(0 \text{ or } 1 \text{ according as } n \text{ is even}\overset{\text{or}}{\wedge}\text{odd})+\log_e(1+1)$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\, S_n = \log_e 2 \quad \text{or} \quad 1+\log_e 2$$

according as n is even or odd.

Hence by definition the given series is oscillating.

Note—In all the examples to follow, the notation Lt u_n shall stand for $\underset{n\to\infty}{\text{Lt}}\, u_n$

EXERCISE No. 4

Test the convergency of the following series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\ldots\ldots$$

we can write the given series as

$(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+\ldots\ldots$

Combining even and odd brackets separetely

$=(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots)+(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots\ldots)$

The series in each brackets are clearly convergent by Art. 6 and hence the given series is also convergent.

[For a simpler method see Ex. 2 of Art 10]

2 $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\ldots\ldots$ (Convergent)

Proceed exactly as in solved example 1 Page 163.

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\ldots$**(Convergent)**

Proceed as above. **(Andhra 37, L.C.S. 43)**

4. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}\ldots\ldots$**(Convergent)**

u_n is numerically equal to $(\frac{1}{2})^{n-1}$

$\therefore$ Lt $u_n=0$ Hence as before the series is convergent

5. $\frac{1}{x}-\frac{1}{x+a}+\frac{1}{x+2a}-\frac{1}{x+3a}+\ldots\ldots x$ and a being +ive

Proceed as above **(Convergent)**

6. $\frac{1}{xy}-\frac{1}{(x+1)(y+1)}+\frac{1}{(x+2)(y+2)}-\frac{1}{(x+3)(y+3)}+\ldots\ldots$

(Con.)

u_n is numerically equal to

$$\frac{1}{(x+n-1)(y+n-1)}$$

$\therefore$ Lt $u_n=0$, hence as before the series is convergent.

7. $1-2x+3x^2-4x^3+\ldots\ldots$ where x is less than 1.

(Con)

u_n is numerically equal to nx^{n-1}

Lt $u_n=$Lt $nx^{n-1}=x$ Lt $nx^n=0$ by 4·6 Page 160

Hence as before the series is convergent

8. $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\ldots\ldots$ [Convergent]

9. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots$ [Convergent]

***10. Show that the series** (Pb 43)

$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{1}{8}-\frac{1}{9}-\ldots\ldots$

in which the first term is +ive, the next two −ive, the next four +ive, the next eight −ive etc, is not convergent.

If the series be convergent, then the series obtained by grouping the terms should also be convergent i. e.

$1-(\frac{1}{2}+\frac{1}{3})+(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7})-(\frac{1}{8}+\frac{1}{9}+\ldots)+\ldots$ is convergent

$\frac{1}{2}+\frac{1}{3}>\frac{1}{4}+\frac{1}{4}$ i. e. $>2\cdot\frac{1}{4}$ or $>\frac{1}{2}$

Similarly $\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}>4\cdot\frac{1}{8}$ i. e $>\frac{1}{2}$

Thus every term of the grouped series is greater then $\frac{1}{2}$. Hence the above series even though having its terms alternately +ive & −ive but its nth term does not tend to zero & as such it is not convergent and hence the given series is also not convergent.

§7 (a) A series is convergent, if from and after certain fixed term, all its terms are less than the corresponding terms of a second series which is known to be convergent. **[Result]**

Proof Let the two series be

$$u_1+u_2+u_3+u_4+v_5+\ldots\ldots u_n+\ldots\ldots$$

and $$v_1+v_2+v_3+v_4+v_5+\ldots\ldots v_n+\ldots\ldots$$

where the later series is known to be convergent.

Now let us suppose that from and after the 3rd term each term of the series is less than the corresponding term of the second series

$\therefore\quad u_3+u_4+u_5+\ldots\ldots u_n < v_3+v_4+v_5+\ldots\ldots v_n$

$\because\quad \text{Lt}\,(u_3+u_4+u_5+\ldots\ldots u_n) \leqslant \text{Lt}\,(v_3+v_4+v_5+\ldots\ldots v_n)$

Now if $v_1+v_2+v_3+\ldots\ldots v_n+\ldots$ is convergent then by §5 (b) P. 161 $v_3+v_4+v_5+\ldots\ldots v_n+\ldots$ is also convergent and hence $\text{Lt}\,(v_3+v_4+v_5+\ldots\ldots v_n)$ is finite

$\therefore\quad \text{Lt}\,(u_3+u_4+u_5+\ldots\ldots u_n) \leqslant$ finite quantity i. e. = some finite quantity.

Hence by definition $u_3+u_4+u_5+\ldots u_n+\ldots$ is convergent

Therefore by §5 (b) P. 161 $u_1+u_2+u_3+u_4+u_5+\ldots u_n+\ldots$ is also convergent.

Meaning of "from and after certain fixed term" we have read in article 5 that the nature of a series is unaltered by the addition or removal of finite number of terms or by multiplication of each term by a constant.

Hence if we find that a certain condition laid down for the convergency and divergency of a series does not hold in the first finite number of terms then we can very easily leave those terms without affecting the nature of the series.

7. (b) Similarly a series is divergent if from and after certain fixed term all its terms are greater than the corresponding terms of a series which is known to be divergent. (Result)

Ex. 1. Show that the series

$$\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots$$

is convergent.

$$\frac{1}{1.2.3} < \frac{1}{1.2.2} \text{ or } < \frac{1}{2^2}$$

$$\frac{1}{1.2.3.4} < \frac{1}{1.2.2.2} \text{ or } < \frac{1}{2^3}$$

……

Hence we find that from and after the 3rd term each term of the given series is less than the corresponding terms of the series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots\ldots$$

But later series being a G. P. whose common ratio $\frac{1}{2}$ is less than 1 and hence convergent. [See Page 155]

$\therefore$ The given series by §7 (a) is also convergent.

***Ex. 2. Test the convergency of the series**

$$1+\frac{4}{2\,!}+\frac{4^2}{3\,!}+\frac{4^3}{4\,!}+\frac{4^4}{5\,!}+\frac{4^5}{6\,!}+\frac{4^6}{7\,!}+\frac{4^7}{8\,!}+\ldots\infty$$

$$\frac{4^5}{6\,!}=\frac{4^5}{6.5\,!}<\frac{4^5}{5\;5\,!}$$

$$\frac{4^6}{7\,!}=\frac{4^6}{7.6.5\,!}<\frac{4^6}{5^2.5\,!}$$

$$\frac{4^7}{8\,!}=\frac{4^7}{8.7.6.5\,!}<\frac{4^7}{5^3.5\,!}$$

$$\therefore\ \frac{4^5}{6\,!}+\frac{4^6}{7\,!}+\frac{4^7}{8\,!}+\ldots<\frac{4^5}{5.5\,!}+\frac{4^6}{5^2.5\,!}+\frac{4^7}{5^3.5\,!}+\ldots\ldots$$

But the later series being a G. P. whose common ratio $\frac{4}{5}$ is less than 1 and hence convergent.

Therefore $\frac{4^5}{6\,!}+\frac{4^6}{7\,!}+\frac{4^7}{8\,!}\ldots\ldots$ is also convergent by §5 (a)

Hence the given series is also convergent by §5 (b)

Ex. 3. Show that the series

$$\frac{a+x}{b+x}+\frac{(a+x)(2a+x)}{(b+x)(2b+x)}+\frac{(a+x)(2a+x)(3a+x)}{(b+x)(2b+x)(3b+x)}+\ldots\ldots$$

is convergent if a, b, and x are all +ive and $a < b$

For all integral values of r we have with the given condition $a<b$.

$$\frac{ra+x}{rb+x}<\frac{a+x}{b+x}$$

∴ The terms of the given series from and after the second term are less than the corresponding terms of the series

$$\frac{a+x}{b+x}+\frac{(a+x)^2}{(b+x)^2}+\frac{(a+x)^3}{(b+x)^3}+\ldots\ldots$$

The later series is a G. P. whose common ratio is $\frac{a+x}{b+x}$ which is less than one because $a<b$ and hence it is covergent.

Therefore the given series is also convergent.

***Ex. 4. If the series $u_1^2+u_2^2+u_3^2+\ldots\ldots$ is convergent and $v_1^2+v_2^2+v_3^2+\ldots\ldots$ is also convergent, then will $u_1v_1+u_2v_2+u_3v_3+\ldots\ldots$ be also convergent.**

Now $\Sigma\, u_n^2$ and $\Sigma\, v_n^2$ is convergent. ∴ $\frac{1}{2}\Sigma\,(u_n^2+v_n^2)$ is also convergent.

$$\therefore\quad \frac{u_1^2+v_1^2}{2}+\frac{u_2^2+v_2^2}{2}+\frac{u_3^2+v_3^2}{2}\ldots\ldots \text{ is convergent.}$$

Hence $u_1v_1+u_2v_2+u_3v_3+\ldots\ldots$ is also convergent.

$$\therefore\quad u_1v_1<\frac{u_1^2+v_1^2}{2} \text{ and so on}$$

i.e. every term of the second series is less than the corresponding term of the first series which is convergent. Hence the second series is also convergent.

§ 8. A series in which all the terms are of the same sign is divergent if each term is greater than some finite fixed quantity however small. [Result]

$$S_n=u_1+u_2+u_3+\ldots\ldots u_n.$$

∴ $S_n>na$, when each term is greater than a finite quantity a (say).

Now by choosing n sufficiently large we can make S_n exceed any finite quantity and hence by our definition the series is divergent.

Ex. 1. Show that the series given below is divergent

$$1+3+5+7+9+\ldots\ldots$$

In the given series all the terms are +ive and each term is greater than unity and hence by art. 8 the series is divergent.

Ex. 2. $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\ldots\ldots$

All the terms being +ive and each term being greater than 1, the series is divergent.

§ **8·1.** In the above article (a) must be +ive and fixed that is independent of n and each term is to be greater than a for divergency.

***Hence a series of positive terms is divergent if**

$$\text{Lt } u_n > 0. \qquad \text{(Remember)}$$

***§ 9. The infinite series** $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$

i. e. $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots\frac{1}{n^p}+\ldots\ldots$

is convergent if p is +ive and greater than 1 and divergent if p is less than or equal to 1. (Proof)

(Agra 38, 54, Rajputana 50, Pb. 33, 41, 45, 48)

Case 1. If p is +ive and greater than 1.

$$3^p>2^p \quad \therefore \quad \frac{1}{3^p}<\frac{1}{2^p} \quad \therefore \quad \frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p} \text{ or } <\frac{2}{2^p}$$

$$\left\{\begin{array}{l} 5^p>4^p \\ 6^p>4^p \\ 7^p>4^p \end{array}\right\} \quad \therefore \quad \left|\begin{array}{c} \frac{1}{5^p}<\frac{1}{4^p} \\ \frac{1}{6^p}<\frac{1}{4^p} \\ \frac{1}{7^p}<\frac{1}{4^p} \end{array}\right| \quad \therefore \quad \frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{4}{4^p}$$

Similarly $\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\ldots\ldots\frac{1}{15^p}\right)<\frac{8}{8^p}$, and so on.

Hence the given series

$$\frac{1}{1^p}+\left[\frac{1}{2^p}+\frac{1}{3^p}\right]+\left[\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right]$$
$$+\left[\frac{1}{8^p}+\frac{1}{9^p}+\ldots\ldots\frac{1}{15^p}\right]+\ldots\ldots$$

is less then $\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots\ldots$

But the later series is a G. P. whose common ratio $\frac{2}{2^p}$ i. e. $\frac{1}{2^{p-1}}$ is less than 1 when p is +ive and greater than 1. We know that a G. P. whose common ratio is less than 1 is always convergent (See p. 155) and hence later series is convergent and the given series being less than it must therefore also be convergent.

Case II Let p=1

$3<4 \therefore \frac{1}{3}>\frac{1}{4} \therefore \frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}$ or $>\frac{2}{4}$ or $>\frac{1}{2}$

$$\left\{\begin{matrix}5<8\\6<8\\7<8\end{matrix}\right\} \therefore \left|\begin{matrix}\frac{1}{5}>\frac{1}{8}\\[1ex]\frac{1}{6}>\frac{1}{8}\\[1ex]\frac{1}{7}>\frac{1}{8}\end{matrix}\right|$$

$\therefore \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$ or $>\frac{4}{8}$ or $>\frac{1}{2}$

Similarly $\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\ldots\ldots\frac{1}{16}\right)>\frac{8}{16}$ or $>\frac{1}{2}$

Hence the given series

$\frac{1}{1}+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$

$$+(\tfrac{1}{9}+\tfrac{1}{10}+\ldots\ldots\ldots\tfrac{1}{16})+\ldots\ldots$$

is greater than

$$1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\ldots\ldots$$

The later series by art 8 is divergent and the given series being greater than it must therefore also be divergent. §7.b

Case III. p **is less than 1, or −ive.**

In this case $2^p<2$ $\left\{\because \begin{array}{l} 2^{1/2}<2 \\ 2^{-3}<2 \end{array}\right\}$

$$\therefore\ \frac{1}{2^p}>\tfrac{1}{2}$$

Similarly $\dfrac{1}{3^p}>\tfrac{1}{3}$

$\dfrac{1}{4^p}>\tfrac{1}{4}$ and so on

Hence the given series

$\dfrac{1}{1^p}+\dfrac{1}{2^p}+\dfrac{1}{3^p}+\dfrac{1}{4^p}+\ldots\ldots\ldots$ is such that from and after the first term all its terms are greater than the corresponding terms of $1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\ldots$

which in Case II we have proved to be divergent.

Therefore the given series by art 7.(b) is also divergent.

Test the convergency and divergency of the following series.

Ex. 1. $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots\ldots\dfrac{1}{n^2}+\ldots\ldots$

convergent $\because$ $p=2$ i. e. >1

Ex. 2. $\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\ldots\dfrac{1}{n}+\ldots\ldots$

Divergent $\because$ $p=1$

Ex. 3. $\frac{1}{1}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots\dots$

Divergent $\because p=\frac{1}{2}$ i. e. <1

Ex. 4. $\frac{1}{1^{k-1}}+\frac{1}{2^{k-1}}+\frac{1}{3^{k-1}}+\dots\dots\frac{1}{n^{k-1}}+\dots\dots$

Here the given series is convergent if $k-1>1$ i. e. if $k>2$ and divergent if $k-1 \leqslant 1$ i. e. $k \leqslant 2$.

Ex. 5. $1+\frac{1}{2 \, . \, 2^{1/100}}+\frac{1}{3 \, . \, 3^{1/100}}+\frac{1}{4 \, . \, 4^{1/100}}+\dots\dots$

convergent $\because p=1+\frac{1}{100}$ i. e. >1

Ex. 6. $1+\frac{2^{1/100}}{2}+\frac{3^{1/100}}{3}+\frac{4^{1/100}}{4}+\dots\dots$

divergent $\because p=1-\frac{1}{100}$ i. e. <1

***Ex. 7. Prove that in order that an infinite series may be convergent, it is necessary but not sufficient that the limit of its nth term be zero when n tends to infinity. (Agra 45)**

We have proved in Art. 8·1 Page 170 that a series is divergent if Lt $u_n>0$. Hence it follows that for every convergent series Lt $u_n=0$. The converse of it is not true because if we consider a series.

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots\dots+\frac{1}{\sqrt{n}}+\dots\dots$$

which is clearly divergent as $p=\frac{1}{2}$ even though

$$\text{Lt } u_n=\text{Lt } \frac{1}{\sqrt{n}}=0$$

Hence for convergent series Lt $u_n=0$ is the condition which is necessary but not sufficient.

§10. If there are two series in each of which all the terms are positive and if the ratio of the corresponding terms in the two series is always finite, the two series are either both convergent or both divergent. (Proof)

i. e. If Lt $\frac{u_n}{v_n}$=finite then Σu_n and Σv_n are either both convergent or both divergent.

(Agra 30, 32, 39 Pb. 33)

Proof Let the two series be

$$u_1+u_2+\ldots\ldots u_n+\ldots\ldots$$

and $$v_1+v_2+\ldots\ldots v_n+\ldots\ldots$$

Since $\frac{u_1}{v_1}$, $\frac{u_2}{v_2}$, $\frac{u_3}{v_3}$ are all finite

Let $\frac{u_1}{v_1}$ has the greatest value say x amongst the above fractions

then $\frac{u_1}{v_1}=x \quad \therefore \quad u_1=v_1x$

$\frac{u_2}{v_2}<x \quad \therefore \quad u_2<v_2x$

Similarly $u_3<v_3x$ and so on

$\therefore \quad (u_1+u_2+\ldots\ldots)<(v_1+v_2+\ldots\ldots)x$

i. e. $\frac{(u_1+u_2+\ldots\ldots)}{(v_1+v_2+\ldots\ldots)}<x$.........(A)

Again let $\frac{u_3}{v_3}$ has the least value say y amongst the above fractions.

$\therefore \quad \frac{u_1}{v_1}>y \quad \therefore \quad u_1>v_1y$

$\frac{u_2}{v_2}>y \quad \therefore \quad u_2>v_2y$

$\frac{u_3}{v_3}=y \quad \therefore \quad u_3=v_3y$

and so on

∴ $(u_1+u_2+u_3+\ldots\ldots)>y(v_1+v_2+v_3+\ldots\ldots)$

i. e. $\dfrac{u_1+u_2+u_3+\ldots\ldots}{v_1+v_2+v_3+\ldots\ldots}>y\ldots\ldots B$

Hence by A and B we can say that the value of the fraction $\dfrac{u_1+u_2+u_3+\ldots\ldots}{v_1+v_2+v_3+\ldots\ldots}$ lies between the greatest and least of the fractions $\dfrac{u_1}{v_1}, \dfrac{u_2}{v_2}, \dfrac{u_3}{v_3}, \ldots\ldots$and is therefore a finite quantity L say

∴ $u_1+u_2+u_3+\ldots\ldots=L(v_1+v_2+v_3+\ldots\ldots)$

Now if v series is convergent then $v_1+v_2+v_3+\ldots\ldots$ is finite ∴ $L(v_1+v_2+v_3+\ldots\ldots\ldots)$ is finite and hence $u_1+u_2+u_3+\ldots\;\ldots$is finite i. e. u series is convergent

Similarly if v series is divergent then u series is also divergent :—

Note :—The application of this principle is very important. for by means of it, we can compare a given series with an auxilliary series whose convergency or divergency has already been established. The series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^p}$ will be generally found useful as an auxiliary series. The only thing that we have to observe in the application of above article is that

$\text{Lt}\,\dfrac{u_n}{v_n}=\text{finite}$

Method of finding the auxiliary series :—

Take u_n the nth term of the given series. Find its approximate value by taking n large. Out of the approximate value of u_n, retain only the highest power of n and denote that by v_n which will be the nth term of the auxiliary series and also $\text{Lt}\,\dfrac{u_n}{v_n}$ will be finite because u_n and v_n are of the same degree (4·5).

***Ex. 1. Find whether the series whose general term $u_n=(n^3+1)^{1/3}-n$ is convergent or divergent.**

(Nagpur 31)

Let us find the appoximate value of u_n when n is large

$$u_n=n\left(1+\frac{1}{n^3}\right)^{1/3}-n$$

$$=n\left\{1+\frac{1}{3n^3}\cdots\cdots-1\right\}=\frac{1}{3n^2}$$

Retaining only the highest powers of n we get our auxiliary series whose general term $v_n=\frac{1}{n^2}$.

$\therefore$ $\text{Lt}\,\frac{u_n}{v_n}=\frac{1}{3n^2}\Big/\frac{1}{n^2}=\frac{1}{3}=$ a finite quantity

Hence both series are either convergent or divergent

But $\Sigma\, v_n=\Sigma\,\frac{1}{n^2}$ is convergent, as $p=2$ i. e. >1

Hence the given series is also convergent.

Ex. 2. Show that the series **(Pb 42)**

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots\cdots\textbf{is convergent}$$

$$u_n=\frac{1}{n(n+1)}=\frac{1}{n^2+n}$$

Retaining only the highest powers of n we get the general term of our anxiliarly series as $v_n=\frac{1}{n^2}$

$$\text{Lt}\,\frac{u_n}{v_n}=\text{Lt}\,\frac{n^2}{n^2+n}=1=a \text{ finite quantity.}$$

$\therefore$ Both the series either converge or diverge.

But auxiliary series is convergent as $p>1$ and hence the given series is also convergent.

EXERCISE No. 5

Find whether the series whose general terms are given below are convergent or divergent.

***1.** $u_n=\sqrt{(n^2+1)}-n$. **(Agra 39, 45, 47 Supp., Nag. 59, Sagar 48, Madras 39, 40, Luck. 42, 43 Supp., Travancore 45, Bombay 26, Pb. 44)**

$u_n=1/2n$ approx. Proceed as in solved Ex. 1 page 176.

[Divergent]

***2.** $u_n=\sqrt{(n^4+1)}-\sqrt{(n^4-1)}$

(Pb. 35, Patna 48, Travancore 41)

Proceed as in solved Ex. 1 page 176. **[Convergent]**

3. $\frac{2}{1}+\frac{3}{8}+\frac{4}{27}+\dots\dots$ **[Convergent]**

$$u_n=\frac{n+1}{n^3}=\frac{n\left(1+\frac{1}{n}\right)}{n^3}=\frac{1}{n^2}\left(1+\frac{1}{n}\right)$$

$=1/n^2$ appproximately when n is large.

Hence proceeding as before by comparison with auxiliary series we find that the series is convergent.

***4.** $u_n=\frac{(n+1)^3}{[n^k+(n+2)^k]}$

$$=\frac{n^3\left(1+\frac{1}{n}\right)^3}{n^k\left[1+\left(1+\frac{2}{n}\right)^k\right]}=\frac{1}{n^{k-3}[1+1]}\text{ approximately}$$

when n is large

$$=\frac{1}{2n^{k-3}}$$

Choose $v_n=\frac{1}{n^{k-3}}$

Lt $\frac{u_n}{v_n}=\frac{1}{2}$=a finite quantity

But the auxiliary series is convergent or divergent according as $k-3>$ or $\leqslant 1$

i.e. $k>$ or $\leqslant 4$.

Hence the given series is convergent if $k>4$

and divergent if $k\leqslant 4$

5: (a) $u_n=\sqrt{(n+1)}-\sqrt{(n-1)}$ (b) $u_n=\frac{\sqrt{(n+1)}-\sqrt{n}}{n}$

(Pb. 38)

(a) Here $u_n=\frac{1}{\sqrt{n}}$ and hence as before series is divergent

(b) $u_n=\frac{1}{2n^{3/2}}$ etc. [Convergent]

6. $u_n=\frac{2n(n+3)}{(n+1)(n+4)(3n+5)}$ [Divergent]

Here $u_n=\frac{2}{3n}$ approximately when n is large. Hence proceed as before (Divergent)

7. $u_n=\frac{\sqrt[3]{(2n^2-1)}}{\sqrt[4]{(3n^3+2n+5)}}$

$$=\frac{(2n^2)^{1/3}\sqrt[3]{\left(1-\frac{1}{2n^2}\right)}}{(3n^3)^{1/4}\sqrt[4]{\left(1+\frac{2}{3n^2}+\frac{5}{3n^3}\right)}}$$

$=\frac{2^{1/3}.n^{2/3}}{3^{1/4}.n^{3/4}}$ approximately when n is large

$$=\frac{2^{1/8}}{3^{1/4}}\cdot\frac{1}{n^{3/4-2/3}}=\frac{2^{1/8}}{3^{1/4}}\cdot\frac{1}{n^{1/12}}.$$

Take $v_n=\dfrac{1}{n^{1/12}}$

$$\text{Lt}\ \frac{u_n}{v_n}=\frac{2^{1/8}}{3^{1/4}}=\text{a finite quantity}$$

But the auxilliary series is divergent because $p=\frac{1}{12}$ i. e. < 1. Hence the given series is also divergent.

8. $\dfrac{1}{1.2.3}+\dfrac{3}{2.3.4}+\dfrac{5}{3.4.5}+\ldots$ (Sagar 49)

Here $u_n=\dfrac{2n-1}{n(n+1)(n+2)}=\dfrac{2}{n^2}$ approximately when n is large. Hence proceed as before [Convergent]

*9. $\dfrac{2}{1^p}+\dfrac{3}{2^p}+\dfrac{4}{3^p}+\dfrac{5}{4^p}+\ldots\ldots$ (Agra 45, 50)

$$u_n=\frac{n+1}{n^p}=\frac{n\left(1+\frac{1}{n}\right)}{n^p}=\frac{1}{n^{p-1}}\text{ when } n \text{ is large}$$

Hence proceeding as in Q. 4 series is convergent when $p > 2$ and divergent when $p \leqslant 2$.

*10. $1+\dfrac{1}{2^2}+\dfrac{2^2}{3^3}+\dfrac{3^3}{4^4}+\dfrac{4^4}{5^5}+\ldots\ldots$ (Divergent)

(Nagpur 37, Aligarh 47)

If we neglect the first term which will not affect the nature of the series and hence

$$u_n=\frac{n^n}{(n+1)^{n+1}}=\frac{n^n}{(n+1)(n+1)^n}$$

$$=\frac{n^n}{(n+1)n^n\left(1+\frac{1}{n}\right)^n}=\frac{1}{(n+1)\left(1+\frac{1}{n}\right)^n}$$

$=\frac{1}{(n+1).e}$ approximately because when n is large

$\left(1+\frac{1}{n}\right)^n=e$ approximately

Take $v_n=\frac{1}{n}$

Lt $\frac{u_n}{v_n}=\frac{n}{(n+1)e}=\frac{1}{e}=\frac{1}{2.718}$ i. e. a finite quantity

But the auxilliary series is divergent because $p=1$

Hence the given series is also divergent.

*10 (b) $\sum\limits_{1}^{n}\frac{1}{n^{1+1/n}}$ (Agra 44)

$u_n=\frac{1}{n^{1+1/n}}$ choose $v_n=\frac{1}{n}$ $\therefore$ $\frac{u_n}{v_n}=\frac{1}{n.n^{1/n}}.n=\frac{1}{n^{1/n}}$

$\therefore$ Lt $\frac{u_n}{v_n}=$Lt $\frac{1}{n^{1/n}}=1$ $\because$ If $p=n^{1/n}$ then $\log p=\frac{1}{n}\log n$

or Lt $\log p=$Lt $\frac{1}{n}\log n=0$ (4·3) Page 158

$\therefore$ $p\to1$ and hence Lt $\frac{1}{n^{1/n}}=1$, a finite quantity

Hence the two series converge and diverge together, (Art. 10) but since $v_n=\frac{1}{n}$

$\therefore$ v series is divergent (Art. 9)

and so will be the given series

11. $\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\ldots\ldots$ **[Divergent]**

$$u_n=\sqrt{\left[\frac{n}{(n+1)^3}\right]}=\sqrt{\left[\frac{n}{n^3(1+1/n)^3}\right]}=\sqrt{\left(\frac{1}{n^2}=\right)}\frac{1}{n}$$

when n in large

*12. $u_n=\sqrt{(n^3+1)}-\sqrt{n^3}$ **(Convergent)**

$$=n^{3/2}\left\{\left(1+\frac{1}{n^3}\right)^{\frac{1}{2}}-1\right\}$$

$$n^{3/2}\left\{\left(1+\frac{1}{2n^3}\ldots\ldots\right)-1\right\}$$

$$=n^{3/2}\cdot\frac{1}{2n^3} \text{ approximately when } n \text{ is large}$$

$$\frac{1}{2n^{3-3/2}}=\frac{1}{2n^{3/2}}$$

Hence by comparison with auxiliary series we find that it is convergent.

*13. $u_n=\sin\left(\frac{1}{n}\right)$ **(Agra 42, Delhi 49)**

$$=\frac{1}{n}-\frac{1}{3\,!n^3}+\frac{1}{5\,!n^5}-\ldots\ldots$$

$=\frac{1}{n}$ approximately when n is large. Take $v_n=\frac{1}{n}$

$$\text{Lt}\ \frac{u_n}{v_n}=1=\text{a finite quantity}$$

But the auxiliary series is divergent because $p=1$. Hence the given series is also divergent.

Similarly prove that when $u_n=\cos\frac{1}{n}$ then series

(Allahabad 51)

is convergent $\left[, \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\ldots\right]$

***14.** $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \ldots\ldots$ **(Divergent)**

$$u_n = \frac{n}{1+n\sqrt{(n+1)}} = \frac{n}{1+n.n^{1/2}\left(1+\frac{1}{n}\right)^{1/2}}$$

$$= \frac{n}{1+n^{3/2}} \text{ approximately} = \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}} \text{ nearly}$$

15. $u_n = \left(\frac{1}{n} - \log\frac{n+1}{n}\right)$ **(Convergent)**

$$= \frac{1}{n} - \log\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{n} - \left(\frac{1}{n} - \frac{1}{2n^2} + \ldots\ldots\right)$$

$$= \frac{1}{2n^2} \text{ nearly}$$

$\therefore$ By comparison as before series is convergent

***16** $\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \ldots\ldots$

p and q being +ive numbers.

$u_n = \frac{(n+1)^p}{n^q} = \frac{1}{n^{q-p}}$ approximately when n is large.

Hence as in Q. 4 before, [Convergent if $q > p+1$]
[Divergent if $q \leqslant p+1$]

17. Show that the series

$1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \ldots\ldots$ is convergent

$u_n = \frac{1}{(n^2)^{2/3}} = \frac{1}{n^{4/3}}$ Hence convergent by comparison.

18. $u_n = \frac{1}{n} \sin \frac{1}{n}$ **[Convergent]**

$$u_n = \frac{1}{n}\left(\frac{1}{n} - \frac{1}{3!\,n^3} + \dots\dots\dots\right)$$

$$= \frac{1}{n^2} \text{ approximately when } n \text{ is large}$$

19. $u_n = \sin^2 \frac{1}{n} = \left(\frac{1}{n} - \frac{1}{3!\,n^3} + \dots\dots\dots\right)^2$ **(Convergent]**

$$= \frac{1}{n^2} \text{ approximately etc.}$$

20 $u_n = \tan^{-1} \frac{1}{n}$ **[Divergent]**

By Gregory's series $u_n = \left(\frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \dots\dots\right)$

$$= \frac{1}{n} \text{ approximately etc.}$$

21. $u_n = \cot^{-1}(n) = \tan^{-1}\left(\frac{1}{n}\right)$

and hence by question no. 20, is divergent

22. $u_n = \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$ **[Convergent]**

$$= \frac{1}{\sqrt{n}}\left(\frac{1}{n} + \frac{1}{3} \cdot \frac{1}{n^3} \dots\dots\dots\right)$$

$$= \frac{1}{n^{3/2}} \text{ approximately when } n \text{ is large}$$

23. $\frac{1}{(x-d)^2} + \frac{1}{(x-2d)^2} + \frac{1}{(x-3d)^2} + \dots\dots$ **(Agra 32)**

$$u_n = \frac{1}{(x-nd)^2}$$

Retaining highest powers of n we get the auxiliary series whose general term $v_n = \frac{1}{n^2}$

$$\text{Lt}\,\frac{u_n}{v_n}=\text{Lt}\,\frac{n^2}{(x-nd)^2}=\frac{n^2}{n^2d^2+x^2-2nxd}$$

$$=\frac{1}{d^2}\text{ a finite quantity}$$

But auxiliary series is convergent because $p=2$ i. e. >1. Hence the given series is convergent.

24. $\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}+\ldots\ldots$ [Convergent]

Neglecting the first term and then grouping the terms taking two terms in one group we obtain a new series

$$\frac{2x}{x^2-1^2}+\frac{2x}{x^2-2^2}+\frac{2x}{x^2-3^2}+\ldots\ldots$$

$u_n=\frac{2x}{x^2-n^2}$ Now proceed as in Q. 23.

25. If $a_n=\left(1+\frac{x}{n}\right)e^{-x/n}$

find whether the series whose nth term is a_n-1, is convergent or divergent. (Raj. 56)

$$a_n=\left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}+\frac{1}{2!}\cdot\frac{x^2}{n^2}+\ldots\ldots\right)$$

or $a_n=\left(1-\frac{x^2}{n}+\frac{x^2}{2n^2}-\ldots\ldots\right)$

$\therefore\ u_n=a_n-1=-\frac{x^2}{2n^2}$ etc.

Now use § 10 P. 174, the series is convergent.

***§ 11. Ratio Test or D' Alembert's Test. (Proof)**

(a) An infinite series of positive terms is convergent if from and after certain fixed term the ratio of each term to the preceding term is numerically less than some quantity which is itself numerically less than unity. **(Agra 28, Pb. 32, 44)**

(b) The above series is divergent if the above ratio is greater than or equal to unity.

Proof.

(a) Let the series beginning from the fixed term be denoted by $u_1+u_2+u_3+\dots\dots\dots$

and let $\frac{u_2}{u_1}<r$, $\frac{u_3}{u_2}<r$, $\frac{u_4}{u_3}<r$, and so on

Where $r<1$, then

$u_1+u_2+u_3+\dots\dots\dots$

$$=u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\dots\dots\right]$$

$$=u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\ \frac{u_2}{u_1}+\dots\dots\dots\right]$$

$$<u_1[1+r+r^2+r^3+\dots\dots]$$

or $<\frac{u_1}{1-r}$ which is finite.

$$\left\{\because \text{ Sum of an infinite G.P. whose common ratio is } <r \text{ is } \frac{a}{1-r}\right\}$$

Hence the given series is convergent.

(b) In this case we are given that

$$\frac{u_2}{u_1}\geqslant 1 \qquad \frac{u_3}{u_2}\geqslant 1 \qquad \frac{u_4}{u_3}\geqslant 1$$

$$u_2\geqslant u_1 \qquad u_3\geqslant u_2\geqslant u_1 \qquad u_4\geqslant u_3\geqslant u_2\geqslant u_1$$

and so on,

Hence $u_1+u_2+u_3+\dots\dots u_n\geqslant nu_1$

i.e. $S_n>nu_1$.

Now by taking n larger and larger we can make S_n exceed any finite quantity and hence by definition the series is divergent.

Note :—In the practical application of these two tests, to avoid having to ascertain the particular term after which each term is greater or less than the preceding terms it is convenient to find the limit of $\frac{u_{n+1}}{u_n}$ i. e. limit of the ratio of a general term to its preceding term when n is definitely increased. Let the limit be λ.

i e. $\text{Lt} \frac{u_{n+1}}{u_n} = \lambda$

A.
If $\lambda < 1$ the series is convergent (11 a)
If $\lambda > 1$ the series is divergent.
If $\lambda = 1$ the series is either convergent or divergent. A further test will be required.

or B.
1. $\text{Lt} \frac{u_{n+1}}{u_n} < 1$ then convergent
2. $\text{Lt} \frac{u_{n+1}}{u_n} > 1$ then divergent
3. $\text{Lt} \frac{u_{n+1}}{u_n} = 1$. Test fails and a further test is required

Taking the reciprocal of 1, 2 and 3 and hence changing the sign of inequality we get the condition of convergency and divergency as follows.

C.
1. $\text{Lt} \frac{u_n}{u_{n+1}} > 1$......(C) i. e. convergent
2. $\text{Lt} \frac{u_n}{u_{n+1}} < 1$..... (D) i. e. divergent
3. $\text{Lt} \frac{u_n}{u_{n+1}} = 1$. Test fails and a further test is required

Remember that we shall in all the problems to follow use the result (C) for determining the convergency and divergency of series.

Test the convergency and divergency of the following series.

Ex. 1. $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \text{.........}$

$$u_n = \frac{x^n}{(2n-1)2n} \qquad u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$

$$\text{Lt}\ \frac{u_n}{u_{n+1}}=\text{Lt}\ \frac{(2n+1)(2n+2)}{(2n-1)(2n)}\cdot\frac{1}{x}=\frac{1}{x}.$$

$$\therefore\ \frac{1}{x}>1\ldots\ldots C \quad \text{i. e. } x<1\ldots\ldots C$$

$$\frac{1}{x}<1\ldots D. \qquad \therefore\ x>1\ldots D$$

If $x=1$, then $u_n=\dfrac{1}{2n(2n-1)}=\dfrac{1}{4n^2(1-\frac{1}{2}n)}=\dfrac{1}{4n^2}$ approx.

Choose $v_n=\dfrac{1}{n^2}$ $\therefore$ Lt. $\dfrac{u_n}{v_n}=\frac{1}{4}=a$ finite quantity

$\therefore$ Series is convergent because auxilliary series is convergent in which $p=2$ i. e. >1.

Ex. 2, $1+\dfrac{x}{2}+\dfrac{x^2}{5}+\dfrac{x^3}{10}+\ldots\ldots\dfrac{x^n}{n^2+1}\ldots\ldots$

(Andhra 41, Benares 41, Utkal 48)

On neglecting first term

$$u_n=\frac{x^n}{n^2+1},\ u_{n+1}=\frac{x^{n+1}}{(n+1)^2+1}$$

$$\text{Lt}\ \frac{u_n}{u_{n+1}}=\text{Lt.}\ \frac{n^2+2n+2}{n^2+1}\cdot\frac{1}{x}=\frac{1}{x}$$

$$\therefore\ \frac{1}{x}>1\ C \text{ i. e. } x<1\ C$$

$$\frac{1}{x}<1\ D \text{ i. e. } x>1\ D$$

If $x=1$, then $u_n=\dfrac{1}{n^2+1}$ which is convergent as is clear on comparing with the convergent series $v_n=\dfrac{1}{n^2}$

§12. Cauchy's Radical Test or root Test. (Proof)

An infinite series $u_1+u_2+u_3+\ldots\ldots$ (Agra 37, Pb.41) is convergent or divergent according as

$$\textbf{Lt}\ \sqrt[n]{(u_n)}<1 \quad \textbf{or} \quad >1.$$

Let us compare the given series with an infinite G. P. $r+r^2+r^3+$.........which we know is convergent if $r<1$ and divergent if $r>1$.

Then if Lt $\frac{u_n}{r^n}$ be finite then both the series are either convergent or both divergent, (Art 10)

Let $\frac{u_n}{r^n}=k$ then $u_n=kr^n$

$\therefore \quad \sqrt[n]{(u_n)}=k^{1/n}\,.\,r$

Or Lt $\sqrt[n]{(u_n)}=r$. $\because$ Lt $\frac{1}{n}=0$ $\therefore$ Lt $k^{1/n}=k^0=1$

But if $r<1$ the auxilliary series is convergent and hence given series is also convergent. If $r>1$ the auxilliary series is divergent and hence the given series is also divergent.

Note:—If however Lt $\sqrt[n]{(u_n)}=1$

then test fails and a further test will be required.

***Ex. 1.** $u_n=\left(1-\frac{1}{n}\right)^{n^2}$ **(Convergent)**

$\therefore \quad (u_n)^{1/n}=\left\{\left(1-\frac{1}{n}\right)^{n^2}\right\}^{1/n}$

Lt $(u_n)^{1/n}=$Lt $\left(1-\frac{1}{n}\right)^n=e^{-1}=\frac{1}{e}$

$=\frac{1}{2{\cdot}718}$ i. e. <1 .

Hence the given series is convergent :—

***Ex. 2. Show that the series whose general term is $u_n=\frac{1}{(1+1/n)^{n^2}}$ is convergent**

(Bombay 28, Agra 37)

Note :—Now we shall give in the following exercise such questions in which the working rule will be as follows.

✓1. Either Or

$$\text{Lt}\ \frac{u_n}{u_{n+1}} > 1\ldots\ldots C \qquad\qquad \text{Lt}\ \sqrt[n]{(u_n)} < 1\ldots\ldots C$$

$$< 1\ldots\ldots D \qquad\qquad > 1\ldots\ldots D$$

2. If in the above the Limit comes out to be unity then we shall choose any of the following.

(a) Either method of comparison (Art. 10)

(b) Or Lt $u_n > 0$ then the series is divergent (Art. 8·1)

(c) Or Judge convergency and divergency by definition.

EXERCISE No. 3.

[Before doing these questions look to art. 4·4 and 4 5 for limits]

1. (a) $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\ldots\ldots\frac{(n+1)}{n^3}x^n+\ldots\ldots$ **(Patna 50)**

Ans. $x < 1\ldots(C)$, $x > 1\ldots(D)$ and when $x=1$ then

$u_n=\frac{1}{n^2}$ approximately and hence convergent by Art. 10.

(b) Test the series whose general term is $\frac{n+1}{n^2}.x^n$

Ans. $x < 1\ldots(C)$, $x > 1\ldots(D)$ and when $x=1$, Divergent Art. 10.

✓2. **Show that the series** $1+\frac{2^p}{2\,!}+\frac{3^p}{3\,!}+\frac{4^p}{4\,!}+\ldots\ldots$ **is convergent for all values of p.**

(Agra 33, Rajputana 48, Patna 40)

$$\text{Lt}\ \frac{u_n}{u_{n+1}}=\text{Lt}\ \frac{(n+1)!}{n\,!}\cdot\frac{n^p}{(n+1)^p}$$

$$=\text{Lt}\ \frac{n+1}{\left(1+\frac{1}{n}\right)^p}=\infty \text{ i. e. } > 1 \text{ for all values of } p$$

and hence convergent.

3. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots\ldots\frac{n^2-1}{n^2+1}x^n+\ldots\ldots$

Ans. $x<1\ldots(C)$, $x>1\ldots(D)$ and when $x=1$ then

$\text{Lt}\ u_n=\text{Lt}\ \frac{n^2-1}{n^2+1}=1$ i. e. >0 $\therefore$ divergent §8·1

4. $1+3x+5x^2+7x^3+9x^4+\ldots$ $x<1\ldots(C)$, $x>1\ldots(D)$.

Ans. If $x=1$ the series becomes

$1+3+5+7+9+\ldots\ldots$and hence divergent by Art. 8.

5. $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\ldots\ldots$

$$\text{Lt}\ \frac{u_n}{u_{n+1}}=\text{Lt}\ \frac{x+n+1}{x+n}\cdot\frac{1}{x}=\frac{1}{x}\ .$$

$\therefore$ If $\frac{1}{x}>1\ldots\ldots(C)$ i. e. $x<1\ldots\ldots(C)$

$\frac{1}{x}<1\ldots\ldots(D)$ $\qquad x>1\ldots\ldots(D)$

But if $x=1$

$u_n=\frac{1}{1+n}$ and hence on comparison with $v_n=\frac{1}{n}$

the series is divergent.

*6 (a) $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}$

(Agra 48, Rajputana 48, Pb. 50)

$u_n=\frac{x^{2n-2}}{(n+1)\sqrt{n}}$ $\qquad u_{n+1}=\frac{x^{2n}}{(n+2)\sqrt{(n+1)}}$

$$\text{Lt}\ \frac{u_n}{u_{n+1}}=\frac{(n+2)}{n+1}\cdot\sqrt{\left(\frac{n+1}{n}\right)}\cdot\frac{1}{x^2}=\frac{1}{x^2}$$

$\therefore$ If $\frac{1}{x^2}>1\ldots\ldots(C)$ i. e. $x^2<1\ldots\ldots(C)$

$\frac{1}{x^2}<1\ldots\ldots(D)$ $\qquad x^2>1\ldots\ldots(D)$

But if $x^2=1$

$$u_n=\frac{1}{(n+1)\sqrt{n}}=\frac{1}{\sqrt{n}.n\left(1+\frac{1}{n}\right)}=\frac{1}{n^{3/2}}\text{ approximately}$$

which is convergent by comparison test.

*6. (b) $\frac{1}{1^p}+\frac{x}{3^p}+\frac{x^2}{5^p}+\ldots\ldots\frac{x^n}{(2n+1)^p}+\ldots\ldots$ (Agra 41)

Ans. $x<1$ C, $x>1$ D and when $x=1$, then convergent if $p>1$ and divergent if $p\leq 1$,

7. $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+\ldots\ldots$

Ans. $x<1$ C, $x>1$ D and when $x=1$ then D by Art. 8.

8. $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\ldots\ldots$

neglecting the first term.

$$u_n=\left(\frac{n+1}{n+2}\right)^n x^n$$

$$\text{Lt }(u_n)^{1/n}=\text{Lt }\frac{n+1}{n+2}x=x \quad \therefore \quad x<1\text{ C,} \quad >1\text{ D}$$

$$\text{If } x \quad 1\text{, then } u_n=\left(\frac{n+1}{n+2}\right)^n=\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$$

$$\text{Lt }u_n=\frac{e}{e^2}=\frac{1}{e}=\frac{1}{2{\cdot}718}\text{ i. e. }>0.\text{ Hence divergent }(8{\cdot}1)$$

9. $1^2+2^2x+3^2x^2+4^2x^3+\ldots\ldots$ (Andhra 37)

Ans. $x<1$ C, $x>1$ D and when $x=1$, the series becomes $1^2+2^2+3^2+\ldots\ \ldots$ which is divergent by Art. 8.

10. $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots\ldots$ (Dacca 46)

$$u_n=\frac{2n-1}{n!}, \qquad u_{n+1}=\frac{2n+1}{(n+1)!}$$

$$\text{Lt }\frac{u_n}{u_{n+1}}=\text{Lt }\frac{(n+1)!}{n!}\cdot\frac{2n-1}{2n+1}=\text{Lt }\frac{(n+1)(2n-1)}{(2n+1)}=\infty$$

i. e, >1 $\therefore$ Series is convergent.

11. $\dfrac{1}{1+2}+\dfrac{2}{1+2^2}+\dfrac{3}{1+2^3}+\ldots\ldots$

$$u_n=\frac{n}{1+2^n},\ u_{n+1}=\frac{n+1}{1+2^{n+1}}+\ldots\ldots$$

$$\text{Lt}\,\frac{u_n}{u_{n+1}}=\text{Lt}\,\frac{n}{n+1}\cdot\frac{1+2^{n+1}}{1+2^n}$$

$$=\text{Lt}\,\frac{n}{n+1}\times\text{Lt}\,\frac{\frac{1}{2^{n+1}}+1}{\frac{1}{2^{n+1}}+\frac{1}{2}}$$

$$=1\times\frac{0+1}{0+\frac{1}{2}}=2 \text{ i. e. } >1.$$

$\therefore$ The series is convergent.

***12.** $\dfrac{1}{1+2^{-1}}+\dfrac{2}{1+2^{-2}}+\dfrac{3}{1+2^{-3}}+\ldots\ldots$ **(Andhra 48)**

$$u_n=\frac{n}{1+2^{-n}}=\frac{n.2^n}{2^n+1}$$

$$u_{n+1}=\frac{(n+1)2^{n+1}}{2^{n+1}+1}$$

$$\text{Lt}\,\frac{u_n}{u_{n+1}}=\text{Lt}\frac{n}{n+1}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{2^{n+1}+1}{2^n+1}$$

$$=\text{Lt}\frac{n}{n+1}\cdot\frac{1}{2}\cdot\frac{1+\frac{1}{2^{n+1}}}{\frac{1}{2}+\frac{1}{2^{n+1}}}$$

$$=1.\frac{1}{2}\cdot\frac{1}{\frac{1}{2}}=1$$

No result i.e. test fails.

$$\text{Lt } u_n = \text{Lt } n \cdot \frac{2^n}{2^n+1} = \text{Lt } n \,.\, \text{Lt } \frac{1}{1+1/2^n}$$

$$= 1 \,.\, \text{Lt } n = \infty \text{ i. e. } > 0$$

$\therefore$ The series is divergent (8 . 1)

13. $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{65}} + \ldots\ldots$

$$u_n = \frac{2^{n-1}}{\sqrt{(4^{n-1}+1)}} = \frac{2^{n-1}}{\sqrt{[2^{2(n-1)}+1]}}$$

$$= \frac{2^{n-1}}{2^{n-1}\sqrt{\left(1+\frac{1}{2^{2(n-1)}}\right)}} = \frac{1}{\sqrt{\left(1+\frac{1}{2^{2(n-1)}}\right)}}$$

$\text{Lt } u_n = 1$ i. e. $> 0 \quad \because \text{ Lt } \frac{1}{2^{2(n-1)}} = 0$

$\therefore$ Series is divergent

14. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots\ldots$

neglecting the first term

$u_n = \frac{x^n}{(n+1)^n}$, $\therefore$ $\text{Lt } (u_n)^{1/n} = \text{Lt } \frac{x}{n+1} = 0$ for all values of x and hence convergent (Art 12)

15. $1 + \frac{2^2}{2\,!} + \frac{3^2}{3\,!} + \frac{4^2}{4\,!} + \ldots\ldots$ **(Agra 51)**

$$\text{Lt } \frac{u_n}{u_{n+1}} = \text{Lt } \frac{(n+1)\,!}{n\,!} \cdot \frac{n^2}{(n+1)^2}$$

$= \text{Lt } \frac{n^2}{(n+1)} = \infty$ i. e. > 1 and hence convergent

*16. $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots\ldots$

(Allahabad 41)

$$u_n=\left\{\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right\}^{-n}$$

$$\therefore\quad u_n^{1/n}=\left\{\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right\}^{-1}$$

$$=\left(\frac{n+1}{n}\right)^{-1}\left\{\left(\frac{n+1}{n}\right)^{n}-1\right\}^{-1}$$

$$=\left(1+\frac{1}{n}\right)^{-1}\left\{\left(1+\frac{1}{n}\right)^{n}-1\right\}^{-1}$$

Taking limit of both sides

$$\text{Lt }(u_n)^{1/n}=(1+0)^{-1}\{e-1\}^{-1}$$

$$=(e-1)^{-1}=(2\cdot718-1)^{-1}=(1\cdot718)^{-1}$$

$$=\frac{1}{1\cdot718}\text{ i. e. }<1$$

$\therefore$ Series is convergent.

17. $u_n=\dfrac{n^3+a}{2^n+a}$ **(Pb. 45, Nagpur 48)**

$$u_{n+1}=\frac{(n+1)^3+a}{2^{n+1}+a}$$

$$\text{Lt }\frac{u_n}{u_{n+1}}=\text{Lt }\frac{n^3+a}{(n+1)^3+a}\text{ Lt }\frac{2^{n+1}+a}{2^n+a}$$

$$=1\text{ . Lt }2\text{ . }\frac{2^n+a/2}{2^n+a}$$

$$=1\text{ . }2\text{ . }1=2\text{ i. e. }>1$$

$$\frac{1+\frac{a}{n^3}}{1+\frac{a}{(n+1)^3}}$$

$$1\text{ . }\frac{2+\frac{a}{2^n}}{1+\frac{a}{2^n}}$$

2

$\therefore$ Series is convergent

Note :—$\text{Lt }\dfrac{2^n+a/2}{2^n+a}=1$ by Art. 4.5 (a)

18. $\frac{4}{7}x+\frac{7}{11}x^2+\frac{10}{15}x^3+\frac{13}{19}x^4+\ldots$ **(Annamalai 49)**

Here $u_n=\dfrac{3n+1}{4n+3}\cdot x^n$ and hence proceeding as usual if

$$x<1\text{C}$$

$x>1$...(D) and when $x=1$ then Lt $u_n=$ Lt $\frac{3n+1}{4n+3}=\frac{3}{4}$ i.e >0

∴ Divergent by §8·1

19. Test the convergency of the series (Ag. 47)

(i) $\Sigma\frac{x^n}{n\,!}$ (ii) $\frac{(-1)^n.x^n}{n}$

(i) Lt $\frac{u_n}{u_{n+1}}=\infty$, ∴ series is divergent

(ii) Lt $\frac{u_n}{u_{n+1}}=-\frac{1}{x}=\frac{1}{x}$ numerically

∴ If $\frac{1}{x}>1$......(C) i. e. $x<1$......(C) numerically

<1......(D) >1......(D)

If $x=1$, series becomes $=-(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...)=-\log 2$

∴ In this case it is convergent.

If $x=-1$, $u_n=\frac{(-1)^n.(-1)^n}{n}=\frac{1}{n}$ ∴ Divergent ∵ $p=1$

20. $u_n=\frac{n\,!}{n^n}$ (Ph. 32)

∴ Lt $\frac{u_n}{u_{n+1}}=$ Lt $\frac{n\,!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)\,!}$

$=$ Lt $\left(\frac{n+1}{n}\right)^n=$ Lt $\left(1+\frac{1}{n}\right)^n=e$ i. e. >1

Hence series is convergent.

§13. Comparison of ratios.

(a) If u_n and v_n are the general terms of two finite series in which all the terms are +ive, then u series will be convergent when v is given to be convegent and at the same time from and after certain fixed term

$$\frac{u_{n+1}}{u_n} < \frac{V_{n+1}}{V_n} \qquad \text{(Pb. 34, 46)}$$

(b) u series will be divergent when v series is given to be divergent and at the same time, from and after certain fixed term

$$\frac{u_{n+1}}{u_n} > \frac{V_{n+1}}{V_n} \qquad \text{(Pb. 46)}$$

(a) Proof. Given v convergent.

and $\frac{u_2}{u_1} < \frac{v_2}{v_1}$, $\frac{u_3}{u_2} < \frac{v_3}{v_2}$ and so on

$$\therefore \quad u_1+u_2+u_3+\ldots=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}+\ldots\ldots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\ldots\right)$$

$$<u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\ldots\ldots\right)$$

$$\text{Or} \quad <u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_1}+\frac{v_4}{v_1}+\ldots\ldots\right)$$

$$\text{Or} \quad <\frac{u_1}{v_1}(v_1+v_2+v_3+v_4+\ldots\ldots)$$

But v series is convergent $\therefore$ u series is also convergent

(b) Given u divergent and

$$\frac{u_2}{u_1} > \frac{v_2}{v_1}, \quad \frac{u_3}{u_2} > \frac{v_3}{v_2} \text{ and so on}$$

$$\therefore \quad u_1+u_2+u_3+\ldots=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots\ldots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\ldots\right)$$

$$>u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\ldots\ldots\right)$$

$$\text{Or} \quad >u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_1}+\ldots\ldots\right)$$

$$\text{or } > \frac{u_1}{v_1}(v_1+v_2+v_3+\ldots\ldots)$$

But v series is given to be divergent.

Hence u series is also divergent.

Note.	**Form X**	**Form Y**
(a)	v is convergent and $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ then u is also convergent.	v is convergent and or $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ then u is also convergent.
(b)	v is divergent and $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}$ then u is also divergent	v is divergent and or $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$ then u is also divergent.

We will always take the form Y on the right hand side into consideration.

Application

With the help of this article we shall derive some more formula which will enable us to determine the convergency and divergency of series when

$$\mathrm{Lt}\ \frac{u_n}{u_{n+1}} = 1$$

Article 14 and 15 are both deduced by the help of art. 13.

14. Raabi's Test. (Pb. 36, 40)

The series whose general term is u_n is convergent or divergent according as $\mathrm{Lt}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}>1$ or <1.

Let us compare the given series with an auxiliary series whose general term is $v_n = \frac{1}{n^p}$

Case I.

Suppose v series is convergent and hence p is greater than 1. Then by art. 13 (Y) u series is convergent.

$$\text{If } \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \text{ i. e. } > \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} \text{ i. e. } > \left(\frac{n+1}{n}\right)^p$$

$$\text{or } > \left[1+\frac{1}{n}\right]^p \text{ or } > 1+p.\frac{1}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\ldots\ldots$$

$$\text{or if } n\left[\frac{u_n}{u_{n+1}}-1\right] > p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots$$

$$\text{or if Lt } n\left[\frac{u_n}{u_{n+1}}-1\right] > p$$

But p is itself greater than 1.

∴ u series is convergent if

$$\text{Lt}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} > 1$$

Case II.

Suppose v is divergent and hence $p<1$ or $=1$, then by art. 13 (Y) u series will be divergent.

$$\text{If } \frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}} \text{ i. e. } < \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} \text{ i. e. } < \left(\frac{n+1}{n}\right)^p$$

$$\text{or } < \left(1+\frac{1}{n}\right)^p \text{ or } < 1+p.\frac{1}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\ldots\ldots$$

$$\text{or if } n\left(\frac{u_n}{u_{n+1}}-1\right) < p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots$$

$$\text{or if Lt } n\left(\frac{u_n}{u_{n+1}}-1\right) < p$$

But p is itself less than 1 or equal to 1.

∴ u series is divergent if

$$\text{Lt.}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} < 1$$

***Example 1.**

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+\ldots\ldots$$

Neglect the first term $u_n=\dfrac{3.6.9.12\ldots\ldots\ldots 3n}{(7.10.13\ldots\ldots 3n+4)}.x^n$

$$u_{n+1}=\frac{3.6.9.12\ldots\ldots 3n(3n+3)}{7.10.13\ldots\ldots(3n+4)(3n+7)}\cdot x^{n+1}$$

$$\text{Lt}\ \frac{u_n}{u_{n+1}}=\text{Lt}\ \frac{3n+7}{3n+3}\cdot\frac{1}{x}=\frac{1}{x},$$

$\therefore$ If $\dfrac{1}{x}>1$ C i. e. $x<1$ C

$\dfrac{1}{x}<1$ D i. e. $x>1$ D

But if $x=1$ the above limit becomes 1 and our test fails.

$\dfrac{u_n}{u_{n+1}}=\dfrac{3n+7}{3n+3}$ on putting $x=1$

$$\frac{u_n}{u_{n+1}}-1=\frac{3n+7}{3n+3}-1=\frac{4}{3n+3}$$

$\therefore$ $\text{Lt}\ n\left[\dfrac{u_n}{u_{n+1}}-1\right]=\text{Lt}\ \dfrac{4n}{3n+3}=\dfrac{4}{3}$ i. e. >1

and hence convergent.

$\therefore$ $x\leqslant 1$C and $x>1$D

Note—The student should write down the general term very carefully.

(a) All the factors in the numerator and denominator of any term are contained in the succeeding term and one factor each in numerator and denominator is multiplied. Thus fourth term contains all the factors i. e. $\frac{3.6.9}{7.10.13}$ of the preceding term and one factor 12 in numerator and 16 in the denominator is multiplied.

(b) In order to calculate the last factor of denominator and numerator of any term we observe that the numerators of 1st, 2nd, 3rd, 4th and so on terms have for their last factors the numbers 3, 6, 9, 12

Hence the numerator of nth term will have the nth term of the A. P. whose terms are 3, 6, 9, 12,...as its last factor.

$$\text{i. e. } a+(n-1)d$$
$$=3+(n-1)3=3n$$

Similarly the last factor of the denominator of nth term of the given series is the nth term of A. P. whose terms are

$$7,\ 10,\ 13,\ 16\ldots\ldots$$
$$\text{i. e. } a+(n-1)d$$
$$7+(n-1)3$$
$$=3n+4$$

***15. The series whose general term is u_n is convergent or divergent according as** (Cauchy's Logarithmic

$$\text{Lt } n \log\frac{u_n}{u_{n+1}} > 1 \text{ or } < 1$$

(Agra 34, Pb. 37, 49)

Let us compare the given series with an auxilliary series whose general term $v_n=\frac{1}{n^p}$.

Case I

Suppose v series is convergent i. e. $p>1$, then u series by §13 (Y) Page 197 will be convergent.

$$\text{If } \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \text{ i. e. } > \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} \text{ or } > \left[\frac{n+1}{n}\right]^p$$

$$\text{or } > \left[1+\frac{1}{n}\right]^p$$

$$\text{or if } \log\frac{u_n}{u_{n+1}} > p\log\left[1+\frac{1}{n}\right]$$

or $$>p\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots\right]$$

or if $$n\log\frac{u_n}{u_{n+1}}>p-\frac{p}{2n}+\frac{p}{3n^2}\ldots\ldots$$

or if $$\text{Lt}\; n\log\frac{u_n}{u_{n+1}}>p,$$

But p itself is greater than 1.

$\therefore$ u series will be convergent if

$$\text{Lt}\; n\log\frac{u_n}{u_{n+1}}>1$$

Case II.

Suppose v series is divergent, i. e. $p=1$ or <1, then u series by 13 (Y), page 197 will be divergent.

If $$\frac{u_n}{u_{n+1}}<\frac{v_n}{v_{n+1}} \text{ i. e. } <\frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} \text{ or } <\left[\frac{n+1}{n}\right]^p$$

or $$<\left[1+\frac{1}{n}\right]^p$$

or if $$\log\frac{u_n}{u_{n+1}}<p\log\left[1+\frac{1}{n}\right]$$

or $$<p\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots\right]$$

or if $$n\log\frac{u_n}{u_{n+1}}<p-\frac{p}{2n}+\frac{p}{3n^2}-\ldots\ldots$$

or if $$\text{Lt}\; n\log\frac{u_n}{u_{n+1}}<p$$

But p is itself less than 1 or equal to 1.

$\therefore$ u series is divergent if

$$\text{Lt}\; n\log\frac{u_n}{u_{n+1}}<1$$

**Example.

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\frac{5^5x^5}{5!}+\ldots\ldots$$

(Agra 29, Rangoon 37, Alld. 37, Luck. 41, 42, 45, 46, Delhi 47, U. P. C. S. 47, Pb. 49, Raj. 51)

$$u_n=\frac{n^nx^n}{n!},\ u_{n+1}=\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)!}{n}\cdot\frac{n^n}{(n+1)^{n+1}}\cdot\frac{1}{x}$$

$$=n+1.\frac{n^n}{(n+1)(n+1)^n}\cdot\frac{1}{x}$$

$$=\frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}=\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$$\therefore\ \text{Lt}\ \frac{u_n}{u_{n+1}}=\frac{1}{ex}$$

$$\therefore\ \text{If } \frac{1}{ex}>1\ \text{C} \quad \text{i. e.}\ x<\frac{1}{e}\ \text{C}$$

$$\frac{1}{ex}<1\ \text{D} \quad \text{i. e.}\ x>\frac{1}{e}\ \text{D}$$

If however $x=\frac{1}{e}$ then the above limit is equal to 1 and hence our test fails.

$$\frac{u_n}{u_{n+1}}=\frac{e}{\left(1+\frac{1}{n}\right)^n} \text{ on putting } x=\frac{1}{e}$$

$$\log\frac{u_n}{u_{n+1}}=\log e-n\log\left(1+\frac{1}{n}\right)$$

$$=1-n\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots\right]$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \ldots\ldots$$

$$\therefore \quad n \log \frac{u_n}{u_{n+1}} = \tfrac{1}{2} - \frac{1}{3n} + \ldots\ldots$$

Lt $n \log \dfrac{u_n}{u_{n+1}} = \tfrac{1}{2}$ i. e. <1 ∴ series is Divergent

Hence $x < \dfrac{1}{e}$ - C, $x \geqslant \dfrac{1}{e}$ - D

Note :—In all the questions of exercise no. 7 the working rule will be as follows.

(a) Lt $\dfrac{u_n}{u_{n+1}} > 1$ C and <1 D

If Lt $\dfrac{u_n}{u_{n+1}} = 1$ Test fails, then choose

Either

(b) Lt $n\left[\dfrac{u_n}{u_{n+1}} - 1\right] \begin{matrix} >1 & C \\ <1 & D \end{matrix}$

Or

Lt n Log $\dfrac{u_n}{u_{n+1}} \begin{matrix} >1 & C \\ <1 & D \end{matrix}$

EXERCISE No. 7

[Before doing these questions, look to Art 4.4 and 4·5 for limits].

All the questions of this exercise are important.

Test the convergency and divergency of the following series

***1 (a)** $\dfrac{x}{1} + \dfrac{1}{2} \cdot \dfrac{x^3}{3} + \dfrac{1.3}{2.4} \cdot \dfrac{x^5}{5} + \dfrac{1.3.5}{2.4.6} \cdot \dfrac{x^7}{7} + \ldots\ldots$

(Nagpur 34, Mysore 38, Pb. 40 Allahabad 42,
Agra 44, Rajputana 48

$$u_n = \frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)} \; \frac{x^{2n-1}}{2n-1}$$

$$u_{n+1}=\frac{1.3.5......(2n-3)(2n-1)}{2.4.6......(2n-2)\,.\,2n}\;\frac{x^{2n+1}}{2n+1}$$

$$\frac{u_n}{u_{n+1}}=\frac{2n}{2n-1}\cdot\frac{2n+1}{2n-1}\cdot\frac{1}{x^2}$$

$$\text{Lt}\;\frac{u_n}{u_{n+1}}=\text{Lt}\;\frac{4n^2+2n}{4n^2-4n+1}\cdot\frac{1}{x^2}=\frac{1}{x^2}$$

$\therefore$ If $\frac{1}{x^2}>1$ C i. e. $x^2<1$ C

$\frac{1}{x^2}<1$ D i. e. $x^2>1$ D

If $x^2=1$

$$\frac{u_n}{u_{n+1}}=\frac{4n^2+2n}{4n^2-4n+1}$$

$$\therefore\;\frac{u_n}{u_{n+1}}-1=\frac{4n^2+2n}{4n^2-4n+1}-1=\frac{6n-1}{4n^2-4n+1}$$

$$\text{Lt. } n\left[\frac{u_n}{u_{n+1}}-1\right]=\text{Lt}\;\frac{6n^2-n}{4n^2-4n+1}=\frac{6}{4}=\frac{3}{2}\;\text{ i. e. }>1$$

Hence the series is convergent when $x^2=1$

i. e. $x^2\leqslant 1$ C, $x^2>1$ D

(b) Discuss the convergency of $\Sigma\,\frac{1.3.5......(2n-1)}{2.4.6......2(n+1)}\cdot x^n$

*2. $1+\frac{2x}{2\,!}+\frac{3^2x^2}{3\,!}+\frac{4^3x^3}{4\,!}+\frac{5^4x^4}{5\,!}+......$

(Agra 29, 40, 49 Calcutta 38, Rajputana 53)

$$u_n=n^{n-1}\,\frac{x^{n-1}}{n\,!}\;,\;u_{n+1}=(n+1)^n\cdot\frac{x^n}{(n+1)\,!}$$

$$\frac{u_n}{u_{n+1}}=\frac{n^{n-1}}{(n+1)^n}\cdot\frac{(n+1)\,!}{n\,!}\cdot\frac{1}{x}$$

$$=\frac{n^{n-1}}{n^n\left(1+\frac{1}{n}\right)^n}\cdot(n+1)\cdot\frac{1}{x}$$

$$=\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\left(\frac{n+1}{n}\right)\cdot\frac{1}{x}$$

$$\text{Lt}\,\frac{u_n}{u_{n+1}}=\text{Lt}\,\frac{1}{\left[1+\frac{1}{n}\right]^n}\cdot\left[1+\frac{1}{n}\right]\cdot\frac{1}{x}$$

$$=\frac{1}{e}\cdot 1\cdot\frac{1}{x}=\frac{1}{ex}\quad\text{(Art. 4·4)}$$

$$\therefore\quad\text{If }\frac{1}{ex}>1\text{......(C)}\qquad\text{i. e. }x<\frac{1}{e}\text{......(C)}$$

$$\frac{1}{ex}<1\text{......(D)}\qquad x>\frac{1}{e}\text{......(D)}$$

But if $x=\frac{1}{e}$ the above limit is equal to 1 and the test fails. Now apply

$$\text{Lt}\,n\log\frac{u_n}{u_{n+1}}>1\text{......(C)}\quad<1\text{......(D)}$$

$$\frac{u_n}{u_{n+1}}=\frac{1}{\left[1+\frac{1}{n}\right]^n}\left[1+\frac{1}{n}\right]\cdot e\text{ on putting }x=\frac{1}{e}$$

$$=\frac{e}{\left[1+\frac{1}{n}\right]^{n-1}}$$

$$\therefore\quad n\log\frac{u_n}{u_{n+1}}=n\left[\log e-(n-1)\log\left(1+\frac{1}{n}\right)\right]$$

$$=n\left[1-(n-1)\cdot\left\{\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots\right\}\right]$$

$$= n-(n-1)+\frac{(n-1)}{2n}-\frac{(n-1)}{3n^2}+\ldots\ldots$$

$$= 1+\frac{n-1}{2n}-\frac{n-1}{3n^2}+\ldots\ldots$$

$\text{Lt } n \log \frac{u_n}{u_{n+1}} = 1+\frac{1}{2}=\frac{3}{2}$ i. e. > 1 $\therefore$ Convergent

$\therefore$ when $x=\frac{1}{e}$ the series is convergent

$x \leqslant \frac{1}{e}$ Convergent, $x > \frac{1}{e}$ Divergent

3. $1+\frac{x}{2}+\frac{2\,!}{3^2}x^2+\frac{3\,!}{4^3}x^3+\frac{4\,!}{5^4}x^4+\ldots\ldots$

(Agra 38, 42, Alld. 41, Lucknow 42, Utkal 49)

$$u_n=\frac{(n-1)!\,x^{n-1}}{n^{n-1}} \text{ and } u_{n+1}=\frac{n\,!}{(n+1)^n}\cdot x^n$$

$$\frac{u_n}{u_{n+1}}=\frac{(n-1)\,!}{n\,!}\cdot\frac{(n+1)^n}{n^{n-1}}\cdot\frac{1}{x}=\frac{1}{n}\cdot n^n\cdot\frac{\left[1+\frac{1}{n}\right]^n}{n^{n-1}}\cdot\frac{1}{x}$$

$$=\left[1+\frac{1}{n}\right]^n\cdot\frac{1}{x}$$

$\therefore$ $\text{Lt }\frac{u_n}{u_{n+1}}=\frac{e}{x}$ (Art. 4·4)

$\therefore$ If $\frac{e}{x} > 1$……(C) i. e. $x < e$……(C)

$\frac{e}{x} < 1$……(D) $x > e$……(D)

If $x=e$ the above limit$=1$, hence test fails.

Apply logarithmic test.

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{1}{n}\right]^n\cdot\frac{1}{e}\text{ putting }x=e$$

$$\log\frac{u_n}{u_{n+1}}=n\log\left[1+\frac{1}{n}\right]-\log e$$

$$=n\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots\dots\right]-1$$

$$=1-\frac{1}{2n}+\frac{1}{3n^2}\dots\dots-1$$

$$=-\frac{1}{2n}+\frac{1}{3n^2}\dots\dots$$

$$\therefore\quad n\log\frac{u_n}{u_{n+1}}=-\frac{1}{2}+\frac{1}{3n}-\dots\dots\dots$$

$\text{Lt}\, n\log\frac{u_n}{u_{n+1}}=-\frac{1}{2}$ i. e. <1 $\therefore$ series is divergent

$\therefore\quad x<e$ C $\qquad x>e$ D

$$4.\quad 1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^4}{8}+\frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{2\,.\,4\,.\,6\,.\,8\,.\,10}\cdot\frac{x^6}{12}+\dots\dots\dots$$

Neglecting the first term.

$$u_n=\frac{1\,.\,3\,.\,5\,.\,7\,.\,9\dots\dots(4n-5)(4n-3)}{2\,.\,4\,.\,6\,.\,8\,.\,10\dots\dots(4n-4)(4n-2)}\frac{x^{2n}}{4n}$$

$$u_{n+1}=\frac{1.3.5.7.9\dots\dots(4n-5)(4n-3)\,.\,(4n-1)(4n+1)}{2.4.6.8.10\dots\dots(4n-4)(4n-2)\,.\,4n(4n+2)}\cdot\frac{x^{2n+2}}{4n+4}$$

$$\text{Lt}\frac{u_n}{u_{n+1}}=\text{Lt}\frac{4n(4n+2)}{(4n-1)(4n+1)}\cdot\frac{4n+4}{4n}\cdot\frac{1}{x^2}=\frac{1}{x^2}$$

$\therefore$ If $x^2<1$ C, $x^2>1$ D and if $x^2=1$ then by Rabee's Test (Art. 14) series is convergent.

$$*5.\quad x^2+\frac{2^2}{3.4}x^4+\frac{2^2\,.\,4^2}{3\,.\,4\,.\,5\,.\,6}x^6+\frac{2^2\,.\,4^2\,.\,6^2}{3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8}x^8+\dots\dots$$ (Agra 39)

Neglecting the first term

$$u_n=\frac{2^2\,.\,4^2\,.\,6^2\dots\dots(2n)^2}{3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\dots\dots(2n+1)(2n+2)}\cdot x^{2n+2}$$

$$u_{n+1}=\frac{2^2\,.\,4^2\,.\,6^2\ldots\ldots(2n)^2\,.\,(2n+2)^2x^{2n+4}}{3.4\,.\,5.6\,.\,7.8\ldots\ldots(2n+1)(2n+2)(2n+3)(2n+4)}$$

$$\text{Lt}\,\frac{u_n}{u_{n+1}}=\text{Lt}\,\frac{(2n+3)(2n+4)}{(2n+2)^2}\,.\,\frac{1}{x^2}$$

$$=\text{Lt}\,\frac{4n^2+14n+12}{4n^2+4n+4}\,.\,\frac{1}{x^2}=\frac{1}{x^2}$$

$\therefore$ If $x^2<1$ C, $x^2>1$ D and when $x^2=1$ then by Rabee's Test (Art. 14) series is convergent.

*6. (a) $1+a+\frac{a(a+1)}{1.2}+\frac{a(a+1)(a+2)}{1.2.3}+\ldots\ldots$ (Agra 50)

Neglecting the first term

$$u_n=\frac{a(a+1)(a+2)\ldots\ldots(a+n-1)}{1.2.3\ldots\ldots n}$$

$$u_{n+1}=\frac{a(a+1)(a+2)\ldots\ldots(a+n-1)(a+n)}{1.2.3\ldots\ldots n(n+1)}$$

$$\frac{u_n}{u_{n+1}}=\frac{n+1}{n+a}$$

$\text{Lt}\,\frac{u_n}{u_{n+1}}=1.$ Hence test fails.

Apply Rabee's test.

$$\frac{u_n}{u_{n+1}}-1=\frac{n+1}{n+a}-1=\frac{1-a}{n+a}$$

$$\text{Lt}\,n\left[\frac{u_n}{u_{n+1}}-1\right]\ \text{Lt}\,\frac{n(1-a)}{n+a}=1-a$$

$\therefore$ If $1-a>1$ C i. e. $a<0$ C i. e. a −ive C

$1-a<1$ D $a>0$ D i. e. a +ive D

But if $1-a=1$ i. e. $a=0$ then above limit$=1$ and our test fails. In this case the series on putting $a=0$ becomes

$$1+0+0+0+0+\ldots\ldots$$

and hence clearly convergent because S_n is always 1.

∴ $a \lesssim 0$ C

$a > 0$ D

(b) $\dfrac{a}{a+3}+\dfrac{a(a+2)}{(a+3)(a+5)}x+\dfrac{a(a+2)(a+4)}{(a+3)(a+5)(a+7)}x^2+\ldots\ldots$

$x \leqslant$ C, $x>1$ D **(Agra 1935)**

*7. $\dfrac{1^2}{4^2}+\dfrac{1^2.5^2}{4^2.8^2}+\dfrac{1^2.5^2\,9^2}{4^2.8^2.12^2}+\dfrac{1^2.5^2\,9^2.13^2}{4^2.8^2.12^2.16^2}+\ldots\ldots$

(Aligarh 50)

The series by Rabee's Test is convergent.

*8. $\dfrac{a+x}{1}+\dfrac{(a+2x)^2}{2\,!}+\dfrac{(a+3x)^3}{3\,!}+\ldots\ldots$

(Agra 43, 51, Allahabad 43, Benares 48, Sagar 48, Raj. 52)

$$u_n=\frac{(a+nx)^n}{n\,!},\quad u_{n+1}=\frac{\{a+(n+1)x\}^{n+1}}{(n+1)\,!}$$

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)\,!}{n\,!}\cdot\frac{(a+nx)^n}{\{a+(n+1)x\}^{n+1}}$$

$$=\frac{(n+1)n^n x^n\left(1+\dfrac{a}{nx}\right)^n}{(n+1)^{n+1}x^{n+1}\left\{1+\dfrac{a}{(n+1)x}\right\}^{n+1}}$$

$$=\frac{n^n}{n^n[1+(1/n)]^n}\cdot\frac{1}{x}\cdot\frac{\left\{\left(1+\dfrac{a}{nx}\right)^{nx/a}\right\}^{a/x}}{\left[\left\{1+\dfrac{a}{(n+1)x}\right\}^{(n+1)\,x/a}\right]^{a/x}}$$

$$\text{Lt}\ \frac{u_n}{u_{n+1}}=\frac{1}{e}\cdot\frac{1}{x}\cdot\frac{e^{a/x}}{e^{a/x}} \qquad \text{Art (4.4)}$$

$$=\frac{1}{ex}$$

∴ If $\frac{1}{ex}>1$ C i. e. $x<\frac{1}{e}$ C

$\frac{1}{ex}<1$ D $\qquad x>\frac{1}{e}$ D

If $x=\frac{1}{e}$ then limit in the above case is 1 and our test fails. Apply logarithmic test.

$$\frac{u_n}{u_{n+1}}=\frac{1}{\left(1+\frac{1}{n}\right)^n}\,.\,e\,.\,\frac{\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{ae}{n+1}\right)^{n+1}}$$ on putting $x=\frac{1}{e}$

in A.

$$\log\frac{u_n}{u_{n+1}}=\left\{\log e-n\log\left(1+\frac{1}{n}\right)\right\}$$

$$+\left\{n\log\left(1+\frac{ae}{n}\right)-(n+1)\log\left(1+\frac{ae}{n+1}\right)\right\}$$

$$=1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots\right)$$

$$+\left[n\left\{\frac{ae}{n}-\frac{a^2e^2}{2n^2}+\frac{a^3e^3}{3n^3}-\ldots\right\}-(n+1)\left\{\frac{ae}{n+1}-\frac{a^2e^2}{2(n+1)^2}+\frac{a^3e^3}{3(n+1)^3}-\ldots\ldots\right\}\right]$$

$$=\left(\frac{1}{2n}-\frac{1}{3n^2}+\ldots\ldots\right)+\left\{\left(-\frac{a^2e^2}{2n}+\frac{a^3e^3}{3n^2}\ \ldots\ldots\right)-\left(-\frac{a^2e^2}{2(n+1}+\frac{a^3e^3}{3(n+1)^2}-\ldots\ldots\right)\right\}$$

$$n\log\frac{u_n}{u_{n+1}}=\left(\frac{1}{2}-\frac{1}{3n}+\ldots\ldots\right)-\frac{a^2e^2}{2}+\frac{a^3e^3}{3n}+\frac{na^2e^2}{2(n+1)}-\frac{na^3e^3}{3(n+1)^2}+\ldots\ldots$$

$$\text{Lt}\ n\log\frac{u_n}{u_{n+1}}=\tfrac{1}{2}-\frac{a^2e^2}{2}+\frac{a^2e^2}{2}=\tfrac{1}{2}\ \text{i. e.}\ <1$$

∴ Series is divergent

Hence $x < \frac{1}{e}$ ......C and $x \geqslant \frac{1}{e}$ ......D

*9. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots\ldots$

(Agra 37, 41, Allahabad 40, Utkal 46, Raj. 49)

$u_n=x^{n+1}\{\log(n+1)\}^q,\quad u_{n+1}=x^{n+2}\{\log(n+2)\}^q$

$$\frac{u_n}{u_{n+1}}=\frac{1}{x}\,\frac{\{\log(n+1)\}^q}{\{\log(n+2)^q}=\frac{1}{x}\,\frac{\left\{\log n\left(1+\frac{1}{n}\right)\right\}^q}{\left\{\log n\left(1+\frac{2}{n}\right)\right\}^q}$$

$$=\frac{1}{x}\,\frac{\left\{\log n+\log\left(1+\frac{1}{n}\right)\right\}^q}{\left\{\log n+\log\left(1+\frac{2}{n}\right)\right\}^q}$$

$$=\frac{1}{x}\,\frac{\left\{\log n+\frac{1}{n}\ldots\ldots\right\}^q}{\left\{\log n+\frac{2}{n}\ldots\ldots\right\}^q}=\frac{1}{x}\,\frac{\left\{1+\frac{1}{n\log n}\ldots\ldots\right\}^q}{\left\{1+\frac{2}{n\log n}\ldots\ldots\right\}^q}$$

$$=\frac{1}{x}\,\frac{\left\{1+\frac{q}{n\log n}\ldots\ldots\right\}}{\left\{1+\frac{2q}{n\log n}\ldots\ldots\right\}}$$ On expanding binomially

$$=\frac{1}{x}\left\{1+\frac{q}{n\log n}\ldots\ldots\right\}\left\{1+\frac{2q}{n\log n}\ldots\ldots\right\}^{-1}$$

$$=\frac{1}{x}\left\{1+\frac{q}{n\log n}\ldots\ldots\right\}\left\{1-\frac{2q}{n\log n}\ldots\ldots\right\}$$

On expanding binomially

$$=\frac{1}{x}\left\{1+\frac{q}{n\log n}-\frac{2q}{n\log n}\ldots\ldots\right\}$$

On multiplying the two factors

$$=\frac{1}{x}\left\{1-\frac{q}{n\log n}\ldots\ldots\right\}$$

$$\therefore \quad \text{Lt}\ \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ If $\dfrac{1}{x} > 1 \ldots\ldots C$ i. e. $x < 1 \ldots\ldots C$

$\dfrac{1}{x} < 1 \ldots\ldots D$ i. e. $x > 1 \ldots\ldots D$

If $x=1$ the above limit is 1, $\therefore$ Test fails.

Apply Rabee's Test.

$\dfrac{u_n}{u_{n+1}} = 1 - \dfrac{q}{n \log n}$ On putting $x=1$.

$$\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-q}{n \log n}$$

$$\therefore \quad \text{Lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \text{Lt}\ \frac{-q}{\log n} = 0 \text{ i. e. } < 1$$

$\therefore$ The series is divergent.

$\therefore$ $x < 1 \ldots C$ and $x \geqslant 1 \ldots D$. The result is independent of the value of q and hence holds good for all values of q positive or negative.

10. $\dfrac{1}{(\log 2)^p} + \dfrac{1}{(\log 3)^p} + \dfrac{1}{(\log 4)^p} + \ldots\ldots$

(Agra 46, Allahabad 46)

Proceed as in question No. 9. (Divergent)

§ 16. In our working rule we first evaluate the Lt $\dfrac{u_n}{u_{n+1}}$ and the series is convergent or divergent according as the above limit is greater or less than 1. In case the above limit comes out to be equal to 1 we say our test fails; then we evaluate either

Lt $n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$ or Lt $n \log \dfrac{u_n}{u_{n+1}}$ and the series is convergent or divergent according as the above limit is greater or less than 1.

But if the later limit also comes to be equal to 1, then these two tests will also not determine the nature of the series and we shall have to find out some other tests. In order to discover a further test we shall assume the following test called **Cauchy's Condensation test** and is stated as under.

If $\phi(n)$ is +ive for all positive integral values of n and continually diminishes as n increases and if a be a positive integer greater than 1, then the two series

$$\phi(1)+\phi(2)+\phi(3)+\ldots\ldots\phi(n)+\ldots\ldots\ldots(1)$$

and $$a\phi(a)+a^2\phi(a^2)+a^3\phi(a^3)+\ldots\ldots a^n\phi(a^n)+\ldots\ldots(2)$$

are both convergent or both divergent.

Note—In order to obtain the general term of series (2) from the general term of series (1), we take $\phi(n)$ and in it instead of n we place a^n and multiply by a^n.

Thus 3rd term of series (1) is ϕ (3).

∴ 3rd term of series (2) will be $a^3\phi(a^3)$.

Example.

Consider the series whose $u_n=\frac{1}{n}$

i. e. $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\ldots\frac{1}{n}+\ldots\ldots\ldots$

which we know is divergent.

Now taking $a=2$, we get another series whose general term is $$u_n=2^n.\frac{1}{2^n}$$

i. e. writing a^n for n in general term of given series and multiplying by a^n and here $a=2$.

Giving n the values 1, 2, 3, ... we obtain the second series as

$$1+1+1+1+\ldots\ldots\ldots\ldots$$

which is divergent clearly.

Articles 16, 17 and 18 may be left.

17. The series $1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\ldots\frac{1}{n(\log n)^p}+\ldots\ldots$ is convergent or divergent according as $p>1$ or $p\leqslant 1$.

In order to determine the nature of the above series we shall apply Cauchy's Condensation Test (Art. 16).

$$\phi(n)=\frac{1}{n(\log n)^p}=u_n$$

$$\phi(a^n)=\frac{1}{a^n(\log a^n)^p}$$

$$\therefore \quad a^n\phi(a^n)=a^n\frac{1}{a^n(\log a^n)^p}=\frac{1}{(n\log a)^p}$$

$$=\frac{1}{(\log a)^p}\cdot\frac{1}{n^p}=v_n \text{ say.}$$

Now by article No. 16 the series whose general term is $\phi(n)$ i. e. $\frac{1}{(n\log n)^p}$ is convergent or divergent.

So is the series whose general term is $a^n\phi(a^n)$

$$\text{i. e. } \frac{1}{(\log a)^p}\cdot\frac{1}{n^p}$$

Now $v_n=\frac{1}{(\log a)^p}\cdot\frac{1}{n^p}$

Let $v'_n=\frac{1}{n^p}$

then $\text{Lt}\,\frac{v_n}{v'_n}=\frac{1}{(\log a)^p}=$ finite, and hence

by comparison test v_n and v'_n behave together (Art. 10) but the later series is convergent or divergent according as $p>1$ or $\leqslant 1$ (Art. 9) and hence the series whose general term is v_n is convergent or divergent according as $p>1$ or $\leqslant 1$.

Therefore by 'Cauchy's condensation test', the series whose general term is u_n is convergent or divergent according as $p>1$ or $\leqslant 1$.

§ 18. D' Morgan's and Bertrand's Test.

The series whose general term is u_n is convergent or divergent according as

$$\text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]>1 \text{ or } <1$$

Let us compare the given series with the auxiliary series whose general term $v_n=\frac{1}{n(\log n)^p}$

Case I.

Suppose v series is convergent then $p>1$ (Art. 17), then u series by (13Y) page 197 will be convergent if

$$\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}} \quad \text{i. e.} \quad >\frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p}$$

$$\text{i. e.} \quad >\left(1+\frac{1}{n}\right)\left\{\frac{\log n\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

$$\text{or} \quad >\left(1+\frac{1}{n}\right)\left\{\frac{\log n+\log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

$$\text{or} \quad >\left(1+\frac{1}{n}\right)\left\{\frac{\log n+\frac{1}{n}+\ldots\ldots\ldots}{\log n}\right\}^p$$

$$\text{or} \quad >\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\log n}+\ldots\right)^p$$

$$\text{or} \quad >\left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\log n}+\ldots\right) \text{ when } n \text{ is large}$$

$$\text{or} \quad >1+\frac{1}{n}+\frac{p}{n\log n}\ldots\ldots\ldots\text{(A)}$$

or $$n\left(\frac{u_n}{u_{n+1}}-1\right)>1+\frac{p}{\log n}$$

or $$\text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]>p$$

But p is itself greater than 1.

$\therefore$ If $\text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]>1$

then convergent.

Case II.

Suppose v series is divergent then $p<1$ or $=1$. (Art. 17) then u series by (13Y) will be divergent if

$\frac{u_n}{u_{n+1}}<\frac{v_n}{v_{n+1}}$ and proceeding as in case I we get

$$\text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]<p$$

But p is itself <1 or $=1$.

$\therefore$ If $\text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]$

<1 then divergent.

***Example 1.**

Prove that the series whose general term is u_n is divergent if $n\left(\frac{u_n}{u_{n+1}}-1\right)$ is an algebraic function of n which tends to the limit 1 as n tends to infinity.

Let

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{f(n)}{\phi(n)}$$

Since $\text{Lt}\,\frac{f(n)}{\phi(n)}$ is to be 1. $\therefore$ the highest degree terms in $f(n)$ and $\phi(n)$ should be the same § 4·5 and let us put therefore $\frac{f(n)}{\phi(n)}=\frac{n^k+a_1n^{k-1}+\ldots}{n^k+b_1n^{k-1}+\ldots}$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n^k+a_1n^{k-1}+\ldots\ldots}{n^k+b_1n^{k-1}+\ldots\ldots}$$

Since the limit is 1, we shall have to apply art. 18 to determine the nature of the series

$$n\left(\frac{u_n}{u_{n+1}}-1\right)-1=\frac{(a_1-b_1)n^{k-1}+\ldots\ldots}{n^k+b_1n^{k-1}+\ldots\ldots}$$

$$\therefore \quad \text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]$$

$$=\text{Lt}\,\frac{(a_1-b_1)n^{k-1}+\ldots\ldots}{n^k+b_1n^{k-1}+\ldots\ldots}\log n$$

$$=\text{Lt}\,\frac{(a_1-b_1)n^k+\ldots\ldots}{n^k+b_1n^{k-1}\ldots\ldots}\cdot\frac{\log n}{n}=0 \quad \because \text{Lt}\,\frac{\log n}{n}$$

$$=0 \text{ (Art 4·3)}$$

Now 0 is <1. Hence by art 18 the series is divergent.

***Example 2.**

$$\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}x+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\ldots\ldots \qquad \textbf{(Allahabad 50)}$$

$$u_n=\frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2}x^{n-1}$$

$$u_{n+1}=\frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2.(2n+2)^2}\cdot x^n$$

$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}\cdot\frac{1}{x} \quad \therefore \text{Lt}\,\frac{u_n}{u_{n+1}}=\frac{1}{x}$$

Hence if $\frac{1}{x}>1$ C i. e. $x<1$ C, $\frac{1}{x}<1$ D i. e $x>1$ D

If $x=1$ the above limit is 1. $\therefore$ apply Rabee's Test

$$\frac{u_n}{u_{n+1}}=\frac{4n^2+8n+4}{4n^2+4n+1} \qquad \text{putting } x=1$$

$$\frac{u_n}{u_{n+1}}-1=\frac{4n^2+8n+4}{4n^2+4n+1}-1=\frac{4n+3}{4n^2+4n+1}$$

$$\text{Lt}\,n\left(\frac{u_n}{u_{n+1}}-1\right)=\text{Lt}\,\frac{4n^2+3n}{4n^2+4n+1}=1$$

We can say the test fails. But we find that $n\left(\frac{u_n}{u_{n+1}}-1\right)$ is an algebraic function whose limit is 1 and hence by (Ex. 1 art 18) the series must be divergent.

Or proceeding directly

$$n\left(\frac{u_n}{u_{n+1}}-1\right)-1=\frac{4n^2+3n}{4n^2+4n+1}-1=\frac{-n-1}{4n^2+4n+1}$$

$$\text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=\text{Lt}-\frac{(n+1)}{4n^2+4n+1}\cdot\log n$$

$$=\text{Lt}-\frac{(n^2+n)}{4n^2+4n+1}\therefore\text{Lt}\frac{\log n}{n}=0\because\text{Lt}\frac{\log n}{n}=0\text{ Art. 13}$$

But zero being less than 1 $\therefore$ The series is divergent.

Ex. 3. $1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}+\ldots\ldots$

(Agra 46, Allahabad 47)

Divergent by §18.

§19. The series whose general term is u_n is convergent or divergent according as

$$\underset{n\to\infty}{\text{Lt}}\left[n\log\frac{u_n}{u_{n+1}}-1\right]\log n>1\text{ or }<1$$

Let us compare the given series with a series whose general term $v_n=\frac{1}{n(\log n)^p}$

Case 1.

Suppose v series is convergent then $p>1$ (Art 17)

then u series by (13 Y) Page 197 will be convergent

$$\text{if }\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}\text{ i. e. }>\frac{(n+1)\{\log(n+1)\}^p}{n\{\log n\}^p}$$

Now proceeding exactly as in Case I Art. 18.

we get the above condition as

$$\text{if }\frac{u_n}{u_{n+1}}>1+\frac{1}{n}+\frac{p}{n\log n}\ldots\ldots\ldots\ldots\text{A}$$

or If $\log \dfrac{u_n}{u_{n+1}} > \log \left\{1+\left(\dfrac{1}{n}+\dfrac{p}{n\log n}\right)\right\}$

or $> \left(\dfrac{1}{n}+\dfrac{p}{n\log n}\right) - \ldots\ldots\ldots$ expanding log.

or If $n \log \dfrac{u_n}{u_{n+1}} > 1+\dfrac{p}{\log n}$

or If $\text{Lt}\left(n \log \dfrac{u_n}{u_{n+1}} - 1\right)\log n > p \ldots\ldots \text{B}$

But p is itself greater than 1.

Hence the given series is convergent

If $\text{Lt}\left(n \log \dfrac{u_n}{u_{n+1}} - 1\right)\log n > 1$.

Case II.

Suppose v series is divergent, then $p=1$ or < 1

(Art. 17)

then u series by (13 Y) Page 197 will be divergent if

$\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ and proceeding as above we get the

condition if $\text{Lt}\left(n \log \dfrac{u_n}{u_{n+1}} - 1\right)\log n < p$

But p is itself less than 1 or equal to 1

$\therefore$ The given series is divergent if

$\text{Lt}\left(n \log \dfrac{u_n}{u_{n+1}} - 1\right)\log n < 1$

Note :—In all the questions of Exercise 8 the working rule will be as follows.

(a) Apply D' Alembert's Test

$\text{Lt}\dfrac{u_n}{u_{n+1}} > 1 \ldots\ldots \text{C} \quad < 1 \ldots\ldots \text{D}$

If $\text{Lt}\dfrac{u_n}{u_{n+1}} = 1$, the test fails, then choose

Either

(b) Rabee's Test

$$\text{Lt}\ n\left(\frac{u_n}{u_{n+1}}-1\right) > 1 \ldots\ldots \text{C} \quad < 1 \ldots\ldots \text{D}$$

Or

(c) $\text{Lt}\ n \log \dfrac{u_n}{u_{n+1}} > 1 \ldots\ldots \text{C} \quad < 1 \ldots\ldots \text{D}$

If $\text{Lt}\ n\left(\dfrac{u_n}{u_{n+1}}-1\right)=1$ then Test (b) fails, then apply

(d) D' Morgans and Bertrand's Test.

$$\text{Lt}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] > 1 \ldots\ldots \text{C} \quad < 1 \ldots\ldots \text{D}$$

But without applying the Bertrand's Test we can say the series to be divergent when $n\left(\dfrac{u_n}{u_{n+1}}-1\right)$ is an algebraic function tending to 1 as n tends to infinity as we have proved in Ex. 1 Art. 18.

If $\text{Lt}\ n \log \dfrac{u_n}{u_{n+1}}=1$ then test (c) fails and apply.

(e) $\text{Lt}\left(n \log \dfrac{u_n}{u_{n+1}}-1\right)\log n > 1 \ldots\ldots \text{C} \quad < 1 \ldots\ldots \text{D}$

EXERCISE 8

$$1+\frac{a(1-a)}{1^2}+\frac{(1+a)a(1-a)(2-a)}{1^2.2^2}+\frac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2.2^2.3^2}+\ldots\ldots$$

Neglecting the first term

$$u_n=\frac{(n-1+a)\,.\,(2+a)(1+a)a(1-a)(2-a)(3-a)\ldots..(n-a)}{1^2.2^2.3^2\ldots\ldots n^2}$$

$$u_{n+1}=\frac{(n+a)(n-1+a)\ldots(2+a)(1+a)a(1-a)(2-a)(3-a)\ldots(n-a)(n+1-a)}{1^2.2^2.3^2\ldots\ldots n^2(n+1)^2}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{(n+a)(n+1-a)} = \frac{n^2+2n+1}{n^2+n+a(1-a)}$$

$\text{Lt} \dfrac{u_n}{u_{n+1}} = 1$ $\therefore$ Test fails and now apply Rabee's Test

$$\frac{u_n}{u_{n+1}} - 1 = \frac{n^2+2n+1}{n^2+n+a(1-a)} - 1 = \frac{n+1-a(1-a)}{n^2+n+a(1-a)}$$

$$\text{Lt}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \text{Lt} \frac{n^2+n\{1-a(1-a)\}}{n^2+n+a(1-a)} = 1$$

$\therefore$ Test fails. Apply D' Morgan's and Bertrand's Test.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{-na(1-a) - a(1-a)}{n^2+n+a(1-a)}$$

$$\text{Lt}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right] \log n$$

$$= \text{Lt} \frac{-n^2a(1-a) - na(1-a)}{n^2+n+a(1-a)} \cdot \frac{\log n}{n}$$

$$= 0 \text{ because } \text{Lt} \frac{\log n}{n} = 0 \; (4{\cdot}3)$$

0 being less than 1. Hence the series is divergent.

Note—Since $n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$ is an algebraic function tending to 1 as n tends to ∞ and series will be divergent by Ex. 1 Art. 18 and we need not apply the test.

***2.** $\dfrac{a}{b} + \dfrac{a(a+1)}{b(b+1)} + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots\ldots\ldots$

(Agra 40, 49)

$$u_n = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)}{b(b+1)(b+2)\ldots\ldots(b+n-1)}$$

$$u_{n+1} = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)(a+n)}{b(b+1)(b+2)\ldots\ldots(b+n-1)(b+n)}$$

$$\frac{u_n}{u_{n+1}} = \frac{b+n}{a+n} \text{ i. e. } \frac{n+b}{n+a}$$

$\text{Lt} \dfrac{u_n}{u_{n+1}} = 1$ $\therefore$ Test fails. Apply Rabee's Test.

$$\frac{u_n}{u_{n+1}}-1=\frac{n+b}{n+a}-1=\frac{b-a}{n+a}$$

$$\text{Lt } n\left(\frac{u_n}{u_{n+1}}-1\right)=\text{Lt}\frac{n(b-a)}{n+a}=b-a$$

∴ If $b-a>1$ C and $b-a<1$ D

But if $b-a=1$ the above limit is equal to 1 and our test fails. But in this case

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n}{n+a} \text{ on putting } b-a=1$$

which is an algebraic function tending to 1 as n tends to ∞ and hence the series must be divergent in this case. (Art. 18 Ex. 1).

∴ $b-a>1$ C and $b-a\leqslant 1$ D.

or Applying D' Morgan's and Bertrand's Test.

$$n\left(\frac{u_n}{u_{n+1}}-1\right)-1=\frac{n}{n+a}-1=\frac{-a}{n+a}$$

$$\text{Lt}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n=\text{Lt}\frac{-a}{n+a}.\log n$$

$$=\text{Lt}\frac{-an}{n+a}.\frac{\log n}{n}=0 \quad \because \text{ Lt}\frac{\log n}{n}=0 \quad (4\cdot3)$$

Zero being less than 1. ∴ Series is divergent in this case.

*3. (a) $1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2$

$$+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\ldots$$

(Allahabad 27, 36, 39, Bombay 27, Utkal 46, Sagar 49)

Neglecting the first term.

$$u_n=\frac{\alpha(\alpha+1)\ldots\ldots(\alpha+n-1)\beta(\beta+1)\ldots\ldots(\beta+n-1)}{1.2\ldots\ldots n.\gamma(\gamma+1)\ldots\ldots(\gamma+n-1)}x^n$$

$$u_{n+1}=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\ldots(\beta+n-1)(\beta+n)}{1.2\ldots\ldots n(n+1)\gamma(\gamma+1)\ldots\ldots(\gamma+n-1)(\gamma+n)}$$

$$\times x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)} \cdot \frac{1}{x} = \frac{n^2+n(1+\gamma)+\gamma}{n^2+n(\alpha+\beta)+\alpha\beta} \cdot \frac{1}{x}$$

$$\text{Lt}\ \frac{u_n}{u_{n+1}} = \frac{1}{x} \ \therefore \text{If } \frac{1}{x} > 1 \text{ C i. e. } x < 1 \text{ C}$$

$$\frac{1}{x} < 1 \text{ D} \qquad x > 1 \text{ D}$$

If $x=1$ the above limit $=1$ and test fails and we shall apply Rabee's Test.

$$\frac{u_n}{u_{n+1}} = \frac{n^2+n(1+\gamma)+\gamma}{n^2+n(\alpha+\beta)+\alpha\beta} \quad \text{On putting } x=1$$

$$\therefore \frac{u_n}{u_{n+1}} - 1 = \frac{n^2+n(1+\gamma)+\gamma}{n^2+n(\alpha+\beta)+\alpha\beta} - 1$$

$$= \frac{n(1+\gamma-\alpha-\beta)+(\gamma-\alpha\beta)}{n^2+n(\alpha+\beta)+\alpha\beta}$$

$$\text{Lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \text{Lt}\ \frac{n^2(1+\gamma-\alpha-\beta)+(\gamma-\alpha\beta)}{n^2+n(\alpha+\beta)+\alpha\beta}$$

$$= 1+\gamma-\alpha-\beta$$

$\therefore$ If $1+\gamma-\alpha-\beta>1$ C i. e. $\gamma-\alpha-\beta>0$ C

$1+\gamma-\alpha-\beta<1$ D i. e. $\gamma-\alpha-\beta<0$ D

i. e. $\gamma>\alpha+\beta$ C and $\gamma<\alpha+\beta$ D

But if $\gamma=\alpha+\beta$ the above limit becomes 1, and

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] \text{ becomes} = \frac{n^2+n(\gamma-\alpha\beta)}{n^2+n(\alpha+\beta)+\alpha\beta}$$

which is an algebraic function of n whose limit when n tends to ∞ is 1

$\therefore$ The series must be divergent in this case

(Art. 18 Ex. 1)

$\therefore$ Results are $x<1$ C and when $x=1$ then

$x>1$ D convergent if $\gamma>\alpha+\beta$

divergent if $\gamma\leqslant\alpha+\beta$

$$3\text{ (b) } 1+\frac{\alpha}{1.\beta}x+\frac{\alpha(\alpha+1)^2}{1.2.\beta(\beta+1)}.x^2+\frac{\alpha(\alpha+1)^2(\alpha+2)^2}{1.2.3.\beta(\beta+1)(\beta+2)}x^3 + \ldots\ldots\ldots$$

Result $x<1$ C $x>1$ D

and when $x=1$, then convergent if $\beta>2\alpha$

& divergent if $\beta\leqslant 2\alpha$ (Pb 48)

(c) $\dfrac{(1+a)(1+b)}{1.2.3}+\dfrac{(2+a)(2+b)}{2.3\,4}+\ldots+\dfrac{(n+a)(n+b)}{n(n+1)(n+2)}+\ldots$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}}=\frac{(n+a)(n+b)(n+3)}{n(n+a+1)(n+b+1)}$$

$$=\text{Lt}\ \frac{n^3+n^2(a+b+3)+n(ab+3b+3a)+3ab}{n^3+n^2(a+b+2)+n(ab+a+b+1)+1}=1$$

Again $n\left(\dfrac{u_n}{u_{n+1}}-1\right)=\dfrac{n\{n^2+n(2a+2b-1)+3ab\}}{n^3+n^2(a+b+2)+n(ab+a+b+1)+1}$

Now $\text{Lt}\ n\left(\dfrac{u_n}{n_{n+1}}-1\right)$ is clearly 1 and hence by Ex 1. P. 216 or by § 18 the series is divergent.

(d) Test for convergency of the series

$x+x^{1+\frac{1}{2}}+x^{1+\frac{1}{2}+\frac{1}{3}}+x^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}+\ldots\ldots$

(Bombay 27, Luck. 38, Suppl. Agra 34)

$$\frac{u_n}{u_{n+1}}=\frac{x^{1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots 1/n}}{x^{1+\frac{1}{2}+\frac{1}{3}+\cdots(1/n)+1/(n+1)}}=x^{-1/(n+1)}=1\ldots\ldots(1)$$

$\therefore\ \text{Lt}\ \dfrac{u_n}{u_{n+1}}=1$ & hence no result

$$n\log\frac{u_n}{u_{n+1}}=n\,.\left(-\frac{1}{n+1}\right)\log x=\frac{n}{n+1}\,.\log\frac{1}{x}$$

$\therefore\ \text{Lt}\ n\log\dfrac{u_n}{u_{n+1}}=\log\dfrac{1}{x}$

$\therefore$ If $\log\dfrac{1}{x}>1$ C, <1 then D

or If $\log\dfrac{1}{x}>\log e$ C, & $<\log e$ then D

i. e. $\dfrac{1}{x}>e$ C, & $<e$ then D, i.e. $x<\dfrac{1}{e}$ C & $>\dfrac{1}{e}$ D

In case $\log\dfrac{1}{x}=1$ i. e. $x=\dfrac{1}{e}$ then above limit is unity and our test fails

Putting $x=\frac{1}{e}$ in the given series we get from 1

$$\frac{u_n}{u_{n+1}}=\left(\frac{1}{e}\right)^{-1/n+1}=e^{1/(n+1)}$$ and its limit is unity

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=-n(e^{1/(n+1)}-1)=n\left(1+\frac{1}{n+1}+\frac{1}{2\,!\,(n+1)^2}+\dots\dots-1\right)$$

$$=\frac{n}{n+1}+\frac{n}{2\,!\,(n+1)^2}+\dots\dots$$

$\text{Lt}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=1$ and hence no result

Hence by § 18 or Ex. 1 P. 216 the series is divergent

$$\therefore\quad x<\frac{1}{e}\dots\dots\text{C and } x\geqslant\frac{1}{e}\dots\dots\text{D}$$

4. If $$\frac{u_n}{u_{n+1}}=\frac{n^k+An^{k-1}+Bn^{k-2}+Cn^{k-3}+\dots\dots}{n^k+an^{k-1}+bn^{k-2}+cn^{k-3}+\dots\dots}$$

(Allahabad 43)

Where k is a +ive integer, show that the series $u_1+u_2+\dots\dots u_n+\dots\dots$ is convergent if $A-a-1$ is +ive and divergent if $A-a-1$ is −ive or zero.

From above we find that $\text{Lt}\ \frac{u_n}{u_{n+1}}=1$. (Art. 4·5)

∴ Our test fails and we shall now apply Rabee's Test.

$$\frac{u_n}{u_{n+1}}-1=\frac{(A\quad a)n^{k-1}+Bn^{k-2}+Cn^{k-3}+\dots\dots}{n^k+an^{k-1}+bn^{k-2}+cn^{k-3}+\dots\dots}$$

$$\text{Lt}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=\text{Lt}\ \frac{(A-a)n^k+Bn^{k-1}+Cn^{k-2}+\dots}{n^k+an^{k-1}+bn^{k-2}+cn^{k-3}\dots}$$

$$=A-a \qquad \text{(Art. 4·5)}$$

∴ If $A-a>1$...C i. e. $A-a-1>0$ i. e. +ive...C

$$A-a<1 \ldots D \text{ or } A-a-1<0 \quad \text{i. e. } -\text{ive} \ldots D$$

But when $A-a=1$ i. e. $A-a-1=0$ then the above limit becomes 1 and our test fails. But in this case on putting $A-a=1$.

$$n\left(\frac{u_n}{u_{n+1}}-1\right) \text{ becomes} = \frac{n^k+Bn^{k-1}+Cn^{k-2}+\dots\dots}{n^k+an^{k-1}+bn^{k-2}+cn^{k-3}+\dots}$$

which tends to limit 1 when n tends to ∞ (Art. 4·5) and hence by (Art. 18. Ex. 1) the series must be divergent.

$\therefore$ $A-a-1>0$ i. e. $+$ive then convergent

$A-a-1 \leqslant 0$ i. e. $-$ive or equal to zero then divergent.

***5.** $$1^p+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+\dots\dots$$

(Benares 30, Lucknow 45, 50)

neglecting the first term

$$u_n=\left(\frac{1.3.5\dots\dots(2n-1)}{2.4.6\dots\dots2n}\right)^p$$

$$u_{n+1}=\left(\frac{1\,.\,3\,.\,5\dots\dots(2n-1)(2n+1)}{2\,.\,4\,.\,6\dots\dots2n(2n+2)}\right)^p$$

$$\frac{u_n}{u_{n+1}}=\left(\frac{2n+2}{2n+1}\right)^p=\left\{\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right\}^p$$

$\text{Lt}\,\frac{u_n}{u_{n+1}}=1.$ Hence no result

Apply logarithimic test

$$\log\frac{u_n}{u_{n+1}}=p\left[\log\left(1+\frac{1}{n}\right)-\log\left(1+\frac{1}{2n}\right)\right]$$

$$=p\left[\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots\right)-\left(\frac{1}{2n}-\frac{1}{2}\cdot\frac{1}{4n^2}+\frac{1}{3}\cdot\frac{1}{8n^3}-\dots\right)\right.$$

$$\therefore\; n\log\frac{u_n}{u_{n+1}}=p\left[\left(1-\frac{1}{2n}+\frac{1}{3n^2}-\dots\right)\right.$$

$$-\left(\frac{1}{2}-\frac{1}{8n}+\frac{1}{24n^2}-\cdots\right)\Bigg]$$

$$=p\left(\frac{1}{2}-\frac{3}{8n}+\frac{7}{24n^2}-\cdots\right)$$

$$\text{Lt } n\log\frac{u_n}{u_{n+1}}=\frac{p}{2}$$

$\therefore$ If $\frac{p}{2}>1$ C i. e. $p>2$ then C

$\frac{p}{2}<1$ D i. e. $p<2$ then D

But if $p=2$ the above limit$=1$ and test fails, we shall now apply test of Art. 19.

$$n\log\frac{u_n}{u_{n+1}}=2\left(\frac{1}{2}-\frac{3}{8n}+\frac{7}{24n^2}-\cdots\cdots\right)\text{ on putting } p=2$$

$$=1-\frac{3}{4n}+\frac{7}{12n^2}$$

$$\therefore n\log\frac{u_n}{u_{n+1}}-1=\frac{-3}{4n}+\frac{7}{12n^2}$$

$$\therefore \text{Lt}\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n=\text{Lt}\left(\frac{-3}{4}\frac{\log n}{n}+\frac{7\log n}{12n^2}\cdots\right)$$

$$=0 \quad \because \quad \text{Lt}\frac{\log n}{n}=0 \quad (4\cdot3)$$

But 0 being less than 1 $\therefore$ series is divergent.

Hence $p>2$ C, $p\leqslant 2$ D

Note—The following examples are based on Cauchy's condensation test.

6. $\frac{1}{(\log 2)^p}+\frac{1}{(\log 3)^p}+\cdots\cdots\frac{1}{(\log n)^p}+\cdots\cdots$

$$u_n=\frac{1}{(\log n)^p}=\phi(n)$$

$$v_n = a_n \cdot \frac{1}{(\log a^n)^p} = a^n \phi(a^n)$$

By Cauchy's condensation test u and v series are either both convergent or both divergent.

Now consider $v_n = a^n \cdot \frac{1}{(\log a^n)^p} = \frac{a^n}{(n \log a)^p}$

$$v_n = \frac{a^n}{n^p} \cdot \frac{1}{(\log a)^p}$$

$$v_{n+1} = \frac{a^{n+1}}{(n+1)^p} \cdot \frac{1}{(\log a)^p}$$

$$\frac{v_n}{v_{n+1}} = \left(\frac{n+1}{n}\right)^p \cdot \frac{1}{a}$$

∴ Lt $\frac{v_n}{v_{n+1}} = \frac{1}{a}$ which is less than 1.

Because in the above test a is supposed to be a +ive integer greater than 1 (Art. 10)

Since the limit is less than 1 ∴ v series is divergent and hence u series is also divergent.

7. $\frac{(\log 2)^2}{2^2} + \frac{(\log 3)^2}{3^2} + \ldots \ldots \frac{(\log n)^2}{n^2} + \ldots\ldots$

Taking the first term as $\frac{(\log 1)^2}{1^2}$ i. e. 0 we get

$$u_n = \frac{(\log n)^2}{n^2} = \phi(n)$$

$$v_n = a^n \cdot \frac{(\log a^n)^2}{(a^n)^2} = a^n \phi(a^n)$$

By Cauchy's condensation test u and v series are either both convergent or both divergent.

Consider $v_n = a^n \cdot \frac{(\log a^n)^2}{(a^n)^2} = \frac{(n \log a)^2}{a^n}$

$$= \frac{n^2}{a^n} (\log a)^2$$

Example 6, 7, 8 may be left.

$$v_{n+1}=\frac{(n+1)^2}{a^{n+1}}(\log a)^2$$

$$\frac{v_n}{v_{n+1}}=\frac{n^2}{(n+1)^2}\cdot a$$

$\text{Lt}\,\frac{v_n}{v_{n+1}}=a$ which is greater than 1

because in the above test a is supposed to be a +ive integer greater than 1 (Art 16)

Since the limit is greater than 1 $\therefore$ v series is convergent and hence the given u series is also convergent.

8. $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots\frac{1}{n^p}+\ldots\ldots$

$$u_n=\frac{1}{n^p}=\phi(n)$$

$$v_n=a^n\cdot\frac{1}{(a^n)^p}=a^n\phi(a^n)$$

By Cauchy's condensation test u and v series are either both convergent or both divergent.

Now $v_n=\frac{1}{(a^n)^{p-1}}=\left(\frac{1}{a^{p-1}}\right)^n$

and a is +ive integer than 1, say$=2$

$$\therefore\quad v_n=\left(\frac{1}{2^{p-1}}\right)^n$$

On putting $n=1, 2, 3\ldots\ldots$we obtain the series

$$\therefore\quad v_1+v_2+v_3+\ldots\ldots$$

$$=\frac{1}{2^{p-1}}=\left(\frac{1}{2^{p-1}}\right)^2+\left(\frac{1}{2^{p-1}}\right)^3+\ldots\ldots$$

which is a G. P. whose common ratio is $\frac{1}{2^{p-1}}$

By ex. of art 2 Page 155 we know that a G. P. is convergent when its common ratio is less than 1 and divergent when its common ratio is greater or equal to 1.

$\frac{1}{2^{p-1}}$ is < 1 If $p-1 > 0$ i. e. $p > 1$

and $\frac{1}{2^{p-1}}$ is > 1 or $=1$ if $p-1<0$ or $=0$ i.e. $p <$ or $=1$

Hence if $p>1$ then v series is convergent

and if $p \leqslant 1$ than v series is divergent.

$\therefore$ u series will also be convergent if $p>1$

and divergent if $p \leqslant 1$

Infinite products.

§20. with the help of convergency and divergency of series we shall be able to determine whether the product of an infinite number of factors is finite or not.

Let the product be $u_1 u_2 u_3 \ldots\ldots u_n \ldots\ldots$

Let p be the product of n factors then

$p = u_1 u_2 u_3 \ldots\ldots u_n.$

$\therefore$ $\log p = \log u_1 + \log u_2 + \ldots\ldots \log u_n.$

But $\log u_1 + \log u_2 + \ldots\ldots \log u_n$ is the S_n of a series whose general term say $v_n = \log u_n$

Now if we find that v series converges then S_n must be finite when $n \to \infty$ i. e. $\log p =$ finite when $n \to \infty$ and hence $p =$ finite, when $n \to \infty$

Similarly if we find that v series is divergent then

S_n must be either $+\infty$ or $-\infty$ when $n \to \infty$

i. e. $\log p = +\infty$ or $-\infty$ when $n \to \infty$

or $p = -\infty$ or zero when $n \to \infty$

Example 1.

Show that the limit, when n is infinite of

$\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \ldots\ldots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n}$ is finite.

---

§20 and its example may be left.

The product consists of $2n$ factors ; denoting the successive pairs by u_1, u_2......u_n and the above product by p we get

$$p=u_1u_2u_3......u_n$$

where $u_n=\frac{(2n-1)(2n+1)}{2n\,.\,2n}=\frac{4n^2-1}{4n^2}=1-\frac{1}{4n^2}$

$\log p=\log u_1+\log u_2+\ldots.\log u_n$

Now consider a series whose general term

$$v_n=\log u_n=\log\left[1-\frac{1}{4n^2}\right]=-\frac{1}{4n^2}-......$$

On comparing with $v'_n=\frac{1}{n^2}$ (§9)

v series is found to be convergent

$\therefore$ $\log u_1+\log u_2+$ .....$\log u_n=$finite when $n\to\infty$

i. e. $\log p=$finite limit when $n\to\infty$

$\therefore$ p is finite when n is infinite,

Example 2.

$$p=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n}{2n-1}$$

The product consists of $(n-1)$ group of factors each group containing product of 2 factors and one factor $\frac{2n}{2n-1}$ is separate. Denoting the groups by u_1, u_2...and so on.

$$\log p=\log u_1+\log u_2+......\log u_{n-1}+\log\frac{2n}{2n-1}$$

where $u_{n-1}=\frac{(2n-2)(2n-2)}{(2n-3)(2n-1)}$

Now consider a series $v_1+v_2+......v_{n-1}+......$

$=\log u_1+\log u_2+......\log u_{n-1}......$

$\therefore$ $v_{n-1}=\log u_{n-1}=\log\frac{4n^2-8n+4}{4n^2-8n+3}$

$$=\log\left[1+\frac{1}{4n^2-8n+3}\right]$$

$$=\frac{1}{4n^2-8n+3}-\ldots\ldots\text{approximately}$$

choose $v'_{n-1}=\frac{1}{n^2}$

$$\text{Lt}\,\frac{v_{n-1}}{v'_{n-1}}=\text{Lt}\,\frac{n^2}{4n^2-8n+3}=\frac{1}{4}=a\ \text{finite quantity}$$

but v' series is convergent ∵ $p=2$ i. e. >1

∴ v series is also convergent

∴ $\log u_1+\log u_2+\ldots\ldots\log u_{n-1}$ is finite

Hence $\log p=$ finite because,

$$\log\frac{2n}{2n-1}=\log\left[1+\frac{1}{2n-1}\right]=\frac{1}{2n-1}-\ldots\text{approximately}$$

$=0$ when n is large

∴ p is finite.

3. **Show that the nth term tends to zero.**

$$1-\frac{1}{2}+\frac{1.3}{2.4}-\frac{1.3.5}{2.4.6}+\ldots\ldots$$

and hence prove that the series is convergent.

Neglecting the first term

u_n is numerically equal to

$$\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots2n}$$

$\log u_n=\log\frac{1}{2}+\log\frac{3}{4}\ldots\ldots\log\frac{2n-1}{2n}$

Consider a series $v_1+v_2+\ldots\ldots v_n+\ldots\ldots$

where $v_n=\log\frac{2n-1}{2n}=\log\left(1-\frac{1}{2n}\right)=\frac{-1}{2n}-\ldots\ldots$

Choose $v'_n=\frac{1}{n}$

then $\text{Lt}\,\frac{v_n}{v'_n}=\frac{-1}{2}=$ a finite.

$\therefore$ v series is divergent because v' is divergent as $p=1$.

$\therefore$ $v_1+v_2+v_3+\ldots\ldots v_n \to$ infinity.

But v_n is $-$ive.

$\therefore$ $v_1+v_2+v_3+\ldots\ldots v_n \to -\infty$

i. e. $\log u_n \to -\infty$.

$\therefore$ $u_n \to 0$

i. e. limit $u_n=0$.

Now in the above series the terms are alternately $+$ive and $-$ive and each term is less than the preceding term and Lt $u_n=0$. $\therefore$ The series is convergent (Art. 6).

CHAPTER V

DETERMINANTS

§ 1. Consider the following two equations :—

$$a_1x+b_1y=0\text{......}(1)$$
$$a_2x+b_2y=0\text{......}(2)$$
$$\therefore \quad -\frac{a_1}{b_1}=\frac{y}{x}=-\frac{a_2}{b_2}$$

Eliminating x and y, we get $-\frac{a_1}{b_1}=-\frac{a_2}{b_2}$

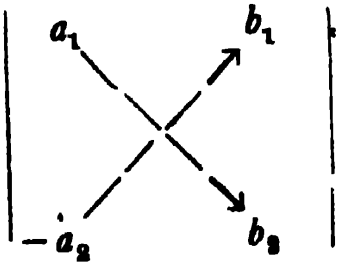
or $\quad a_1b_2-b_1a_2=0.$

We shall express the above eliminant in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0\text{......(A)}$$

i. e. Suppress the letters to be eliminated in the given equations and enclose their coefficients as above in two parallel lines to give you the eliminant.

(A) is called a **Determinant** of second order and its value as we have seen is $a_1b_2-a_2b_1$.



§ 2. Let us nuw eliminate x, y, z from the the following three equations.

$$a_1x+b_1y+c_1z=0\text{......}(1)$$
$$a_2x+b_2y+c_2z=0\text{......}(2)$$
$$a_3x+b_3y+c_3z=0\text{......}(3)$$

Solving equations (2) and (3) by cross-multiplication,

$$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{-(a_2c_3-a_3c_2)}=\frac{z}{a_2b_3-a_3b_2}=k$$

Substituting the values of x, y, z in (1), we get

$$k[a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)]=0.$$

∴ The eliminant is

$$a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)=0.$$

∴ The eliminant shall be written in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \ldots\ldots (A)$$

i. e. suppress the letters to be eliminated in the given equations and enclose their coefficients as above in two parallel lines to give you the eliminant.

A is called a determinant of 3rd order and its value as seen above is

$$a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)=0$$

$$\text{or} \quad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

§ 3. The value of a determinant is called its expansion. We find from above that a determinant of third order has three rows and three columns. It consists of 9 i. e. 3^2 letters which are called constituents and its expansion has 6 i. e. 3! products of the form $a_1b_2c_3$, $-b_1a_2c_3$ and so on which are called elements of the determinant.

Similarly a determinant of nth order shall have n columns, n rows, n^2 constituents and n! elements in it.

§ 4 Leading or principal term.

In determinant (A) the element $a_1b_2c_3$ is called the principal element or term. In this element the suffixes of the constituents occur in their natural order.

Multiply the constituents diagonally starting from the left hand top corner to right hand bottom corner. The term thus obtained will be the principal term.

§ 5, **Expansion of a determinant**—We have seen that

$a_1 \; b_1 \; c_1$ — row

$a_2 \; b_2 \; c_2$ } column

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \; \ldots\ldots(A)$$

$$\text{or} \quad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

The above is called expansion with 1st row,

or a_1(determinant obtained by removing the row and column intersecting at a_1)

$-b_1$(determinant obtained by removing the row and column intersecting at b_1)

$+c_1$(determinant obtained by removing the row and column intersecting at c_1)

Now by rearranging the terms of (A), we find that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

which is called expansion with 1st column. The expansion with 1st column is obtained in the same way as the expansion with 1st row explained before.

Similarly if we have got a determinant of fourth order

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

then following the same law of expansion we can write it as with 1st row,

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

Or expanding with, 1st column the above determinant is equal to

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Note —The above rule of expansion is true when the determinant is expanded with either first row or first column. We will always expand the determinant with either 1st row or 1st column. Expansion can however be made with any other row or column provided such row or column is made the 1st row or column as will be explained further in art. 9.

§ 6. **The value of a determinant is not altered by changing the rows into columns, and the columns into rows.**

(Proof)

$$\text{Lt } \triangle = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2)$$

$$+ c_1(a_2b_3 - a_3b_2) \ldots\ldots \text{A}$$

Let a new determinant Δ' be formed by changing the rows into columns and columns into rows of Δ.

$$\therefore \quad \Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \ldots\ldots \text{B.}$$

Now after re-arranging the terms of B it can be put as $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$.
which is same as A. $\therefore \quad \Delta' = \Delta$

§7 (a) If two adjacent rows or columns of a determinant are interchanged, the sign of the determinant is changed, whereas its numerical value remains the same. (Proof)

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \ldots\ldots \text{A}$$

Let a new determinant Δ' be formed by interchanging the 1st and 2nd rows of Δ.

$$\therefore \quad \Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_2(b_1c_3 - b_3c_1) - b_2(a_1c_3 - a_3c_1) + c_2(a_1b_3 - a_3b_1) \ldots\ldots \text{B}$$

Rearranging the terms of B, it can be put in the form

$$-\{a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\}$$

which is same as A except being of opposite sign

$$\therefore \quad \Delta' = -\Delta.$$

(b) If any two rows or two columns of a determinant are interchanged, the determinant retains its absolute value but changes sign.

Let $\Delta = \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & \end{array} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ......A

Let a new determinant Δ' be formed by interchanging the first and third rows.

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = a_3(b_2c_1 - b_1c_2) - b_3(a_2c_1 - a_1c_2) + c_3(a_2b_1 - a_1b_2) \text{......B}$$

Re-arranging the terms of B it can be put in the form $-\{a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\}$ which is same as A except being of opposite sign $\therefore \quad \Delta' = -\Delta$

. **If any line of a determinant be passed over three parallel lines, then the resulting determinant will be $(-1)^3$ of the original determinant.**

Lt $\Delta = \begin{array}{cccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & & \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{array}$

Let a new determinant Δ' be formed by making the fourth row of Δ the 1st row of Δ' i. e. by making it cross three parallel lines, then we shall prove that $\Delta' = (-1)^3 \Delta$

$$\Delta' = \begin{vmatrix} a_4 & b_4 & c_4 & d_4 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

By reapeated application of Art 7 (a)

$$\Delta'=(-1)\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_4 & b_4 & c_4 & d_4 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

$$\text{or } \Delta'=(-1)^2\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

$$\Delta'=(-1)^3\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}=(-1)^3\Delta$$

Generally if any line of a determinant be passed over p parallel lines, the resulting determinant

$$\Delta'=(-1)^p.\Delta$$

§9. We have done in article 5 (note) that a determinant can be expanded only with first row or first column, but now with the help of article 8 we shall be able to expand the determinant with any row or any column, but we shall have to take into consideration the number of movements of parallel lines which will make that row or column, the first row or column.

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If we want to expand it with 3rd column, then we shall make the third column take the position of 1st column for which we shall have to pass it over 2 columns and hence by article 3

$$\Delta=(-1)^2\begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}.$$

$$=(-1)^2[c_1(a_2b_3-a_3b_2)-c_2(a_1b_3-a_3b_1)+c_3(a_1b_2-a_2b_1)]$$

$$=(-1)^2[a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)]$$

$$=(-1)^2\Delta$$

Similarly expansion with 2nd column will be equal to $(-1)^1\Delta$.

Hence we need not undergo the procedure to bring the row or column with which we want to expand, in the position of first row or column. We may expand with any row or column in accordance with the rules of article 5 but multiply with $(-1)^p$ where p is the number of crossings of lines which will make the expanding row or column 1st row or column.

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ By expanding with 3rd row

$$=(-1)^2[a_3(b_1c_2-b_2c_1)-b_3(a_1c_2-a_2c_1)+c_3(a_1b_2-a_2b_1)]$$

where 2 is the number of crossings of lines which will make the third row take the position of 1st row.

Above can also be written as

$$(-1)^1[b_1(a_2c_3-a_3c_2)-b_2(a_1c_3-a_3c_1)+b_3(a_1c_2-a_2c_1)]$$

***10. If any two rows or two columns of a determinant are identical then the determinant vanishes.**

(Pb. 35, 39)

Let a determinant $\triangle$ be such in which 2nd and 3rd columns are identical.

$$\triangle = \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}$$

Let a new determinant, $\triangle'$ be formed by interchanging the identical columns.

$$\triangle' = \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}$$

The value of $\triangle'$ by article no. 7 must be equal to $-\triangle$. But actually we find that there is no difference between $\triangle'$ and $\triangle$. $\therefore$ $\triangle' = \triangle$ but $\triangle' = -\triangle$ (Art. 7)

Hence $\triangle = -\triangle$ or $2\triangle = 0$ $\therefore$ $\triangle = 0$.

§ 11. Minor.

The determinant that is left by cancelling the row and column intersecting at a particular constituent is called the minor of that constituent and is denoted by the corresponding capital letter.

$$\triangle = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Minor of $b_2 = B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

$$\text{Minor of } c_3 = C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\text{or} \quad a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Both the above forms can be put as

$$\left.\begin{array}{l} \underline{a_1A_1 - b_1B_1 + c_1C_1} \\ \text{or} \quad a_1A_1 - a_2A_2 + a_3A_3 \end{array}\right\} \longrightarrow$$

Expansion in Row

col.

where capital letters denote the minors of the corresponding small letters.

Δ = if expanded with 2nd row or 3rd row.

$$\Delta = (-1)^1 \left\{ a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \right\}$$

$$\text{Or } (-1)^2 \left\{ a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\}$$

Both these can be put as.

$$(-1)^1[a_2A_2 - b_2B_2 + c_2C_2]$$

$$\text{or} \quad (-1)^2[a_3A_3 - b_3B_3 + c_3C_3]$$

$$\therefore \quad a_2A_2 - b_2B_2 + c_2C_2 = -\Delta$$

$$a_3A_3 - b_3B_3 + c_3C_3 = \Delta$$

$$a_1A_1 - b_1B_1 + c_1C_1 = \Delta$$

$$a_1A_1 - a_2A_2 + a_3A_3 = \Delta$$

Hence when the constituents of any row or column are multiplied with their corresponding minors and summed up with alternately plus and minus sign the result $= \Delta$ or $- \Delta$

according as the row or colum whose minors are written is made the first row or column by even or odd crossings of rows or columns.

$$a_1A_1-b_1B_1+c_1C_1=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\triangle$$

but

$$a_2A_1-b_2B_1+c_2C_1=\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{ (Art 10)}$$

Similarly

$$a_1A_1-a_2A_2+a_3A_3=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\triangle$$

but

$$b_1A_1-b_2A_2+b_3A_3=\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}=0 \text{ (Art 10)}$$

Hence if the constituents of any row or column are multiplied with the minors of the constituents of some other row or column and summed up with alternately plus and minus signs the result $=0$

$$\therefore \quad b_1C_1-b_2C_2+b_3C_3=0$$

$$a_3A_1-b_3B_1+c_3C_1=0$$

§12. Co-factor.

The determinant that is left by cancelling the row and column intersecting at a particular constituent, when that particular constituent is brought in the top left hand corner of a determinant is called its co-factor and is denoted by corresponding capital letter.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Co-factor of c_2 : Now c_2 can be brought in top left hand corner by one movement of row and two movements of columns i. e. in all 3 movements.

$$C_2 = \text{co-factor of } c_2 = (-1)^3 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -(\text{minor of } c_2)$$

$$B_2 = \text{co-factor of } b_2 = (-1)^2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = \text{minor of } b_2$$

Hence the co-factors are equal to minors in magnitude but are of same or of opposite sign according as the letter whose co-factor is being calculated is brought to top left hand corner by even or odd movements of lines.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3$$

in terms of minors

$$= a_1A_1 + a_2A_2 + a_3A_3$$

in terms of co-factors.

Just like in article No. 11, it is quite clear that

$$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$$

$$b_1B_1+b_2B_2+b_3B_3=\triangle \qquad \text{(Note)}$$

$$a_3A_3+b_3B_3+c_3C_3=\triangle$$

$$a_2A_2+b_2B_2+c_2C_2=\triangle \qquad \text{(Note)}$$

$$a_1A_2+b_1B_2+c_1C_2=0$$

$$a_3A_1+b_3B_1+c_3C_1=0$$

Where capital letters denote the co-factors of corresponding small letters.

The expansion of a determinant in terms of minors and co-factors differ only in that in the former the terms are alternately plus and minus while in the latter all the terms are plus and that the determinant be expanded with any row or any column the result is either zero or $+\triangle$ (and not $-\triangle$ as in minors).

§13. If each constituent in any row or in any column be multiplied by the same factor, then the determinant is multiplied by that factor.

$$\text{Let } \triangle=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1A_1-b_1B_1+c_1C_1$$

Let a new determinant $\triangle'$ be formed by multiplying the elements of 1st row by a constant m.

$$\triangle'=\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now the minors of ma_1, mb_1 and mc_1 in $\triangle'$ are the same as those of a_1, b_1 and c_1 in $\triangle$

$$\therefore \quad \triangle'=ma_1A_1-mb_1B_1+mc_1C_1$$

$$=m(a_1A_1-b_1B_1+c_1C_1)=m\Delta$$

which proves the proposition.

Note 1—From above it is quite clear that $\Delta=\frac{1}{m}\Delta'$

Hence if in a given determinant Δ the constituents of any row or column be multiplied by a constant m and a determinant Δ' be formed, then

$$\Delta=\frac{1}{m}\Delta'$$

Note 2—From above it is quite clear that if in a determinant the constituents of any row or any column are k times the corresponding constituents of any other row or column, then the determinant vanishes.

i. e.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$$

$\because$ Two rows are identical

§ 14. If each constituent in any row or column consists of two terms then the determinant can be expressed as the sum of two determinants. (Proof)

If A_1, A_2, and A_3 are the minors of a_1, a_2 and a_3 resp. in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then } \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix}$$

$$=(a_1+\alpha_1)A_1-(a_2+\alpha_2)A_2+(a_3+\alpha_3)A_3$$

$$=(a_1A_1-a_2A_2+a_3A_3)+(\alpha_1A_1-\alpha_2A_2+\alpha_3A_3)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Similarly $\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1+\beta_1 & c_1 \\ a_2 & b_2+\beta_2 & c_2 \\ a_3 & b_3+\beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1+\beta_1 & c_1 \\ \alpha_2 & b_2+\beta_2 & c_2 \\ \alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

i. e. sum of four determinants.

We can generalise above as follows :—

If the constituents of the three columns consist of m, n, p terms resp., the determinants can be expressed as the sum of $m.n.p$ determinants.

*§ 15. If to each constituent of a line of determinant be added or subtracted the equi-multiples of the

corresponding constituents of one or more lines, the determinant remains unaltered.

(Punjab 1934, 37, 44, 46, 51)

Let $$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let another determinant Δ' be formed by adding p times and subtracting q times the constituents of 2nd and third column respectively from the corresponding constituents of the first column.

$$\therefore\ \Delta'=\begin{vmatrix} a_1+pb_1-qc_1 & b_1 & c_1 \\ a_2+pb_2-qc_2 & b_2 & c_2 \\ a_3+pb_3-qc_3 & b_3 & c_3 \end{vmatrix}$$

By § 14 it can be broken into three determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix}$$

$$-\begin{vmatrix} qc_1 & b_1 & c_1 \\ qc_2 & b_2 & c_2 \\ qc_3 & b_3 & c_3 \end{vmatrix}$$

Now by §13, the last two determinants vanish.

$$\therefore\ \Delta'=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\Delta$$

In practice generally we can alter several lines of the determinant without altering its value

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1+sc_1 & c_1 \\ a_2+pb_2+qc_2 & b_2+sc_2 & c_2 \\ a_3+pb_3+qc_3 & b_3+sc_3 & c_3 \end{vmatrix}$$

We can split the determinant in the R. H. S. into 3×2 determinants all of which shall vanish except

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

§16. If any rth line has been altered by means of sth and t th parallel lines then sth and tth cannot themselves be altered by means of that rth line, but we can use the rth line to alter all other parallel lines except tth and sth.

$$\text{If} \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Let } \Delta' = \begin{vmatrix} a_1+pb_1+qc_1 & b_1+ra_1+sc_1 & c_1 \\ a_2+pb_2+qc_2 & b_2+ra_2+sc_2 & c_2 \\ a_3+pb_3+qc_3 & b_2+ra_3+sc_3 & c_3 \end{vmatrix}$$

Δ' can be splitted into $3\times 3=9$ determinants. Out of those nine, only seven determinants shall be easily seen to be vanishing and the remaining two will be

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & ra_1 & c_1 \\ pb_2 & ra_2 & c_2 \\ pb_3 & ra_3 & c_3 \end{vmatrix}$$

$$= \Delta - pr\,\Delta = \Delta\,(1 - pr)$$

$\therefore$ $\Delta \neq \Delta'$ because in altering the first column, the second column has been used and again contrary to our article above in altering the second column, the first has been used and hence $\Delta \neq \Delta'$

$$\text{But}\quad \Delta' = \begin{vmatrix} a_1 + pc_1 & b_1 + qa_1 & c_1 \\ a_2 + pc_2 & b_2 + qa_2 & c_2 \\ a_3 + pc_3 & b_3 + qa_3 & c_3 \end{vmatrix}$$

Δ' can be broken into 2×2 i. e. 4 determinants all of which will vanish except

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

$\therefore$ $\Delta' = \Delta$

Here the 2nd column has not been used in altering the first column, though the 2nd has been altered by means of first.

49\5\ **§17. If those constituents of a determinant which involve x, are polynomials in x and if $\Delta = 0$ when a be substituted for x then x—a is a factor of the determinant.**

Now since the constituents of Δ involving x are polynomials in x, then the expansion of Δ will also be a polynomial in x and hence let Δ when expanded.

i. e. $\Delta = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots + p_{n-1}x + p_n$

but $0 = p_0a^n + p_1a^{n-1} + p_2a^{n-2} + \dots\dots + p_{n-1}a + p_n$ (given)

Subtracting

$$\Delta = p_0(x^n - a^n) + p_1(x^{n-1} - a^{n-1}) + p_2(x^{n-2} - a^{n-2}) + \dots\dots + p_{n-1}(x - a)$$

which shows that $x - a$ is a factor of Δ.

Similarly if the constituents of a determinant are in terms of a, b and c, and by putting $a = b$, determinant vanishes i. e. (any of the two parallel lines become identical) then $a - b$ will be a factor of the determinant.

***Example 1. Factorize**

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \text{ or } \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

(Nagpur 30)

Here if we put $a = b$ then the first and second rows become identical and hence $(a - b)$ is a factor. Similarly it is easily seen that $(b - c)$, $(c - a)$ are also its factors.

$\therefore$ $(a - b)(b - c)(c - a)$ is a factor of the determinant. Now the determinant being of 3rd degree as is judged from diagonal term, and the factors that we have found are also of 3rd degree. Hence there can be no other factot in terms of of a, b and c. But a constant can be a factor.

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = K(a - b)(b - c)(c - a) \dots\dots A$$

Now in order to find K, we give a, b and c such small values so that on putting them in the determinant its expansion becomes easier. But care is to be taken in choosing the values, lest the determinant may become zero and the expression on the right also vanishes.

Let $a=0,\ b=1,\ c=-1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = K(0-1)(1+1)(-1)$$

$2=2K \quad \therefore \quad K=1$

$$\text{Hence} \quad \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

The set of values, $a=0,\ b=1,\ c=1$ is not applicable because both L. H. S. as well as R. H. S. vanish.

Note :—Above is the general method of finding the constant factor. The constant factor in the above example could however be found by comparing the coefficients of bc^2 (diagonal term) in both sides of A, and you will find that $K=1$.

***Example 2. Factorize** **(Agra 46, Pb. 50)**

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$ **& find its value when** $a+b+c=0$

Just like in Ex. 1 $(a-b)(b-c)(c-a)$ is easily seen to be a factor of above Now Δ is of 4th degree as is clear from diagonal term and hence we have to search for another factor of first degree because the factors found are of 3rd degree. But Δ is symmetrical and the factors found are symmetrical $\therefore$ The remaining factor must also be symmetrical and of first

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The rusults of Ex. 1 & 2 should be committed to memory.

degree. But a symmetrical factor of first degree in terms of a, b and c can only be $a+b+c$.

$$\therefore\ \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = K(a-b)(b-c)(c-a)(a+b+c)$$

Now the set of values. $\begin{array}{ccc} a & b & c \\ 0 & 1 & -1 \end{array}$ given in previous question is not applicable because R. H. S. vanishes by substituting these values.

Choosing another set of values as $\begin{array}{ccc} a & b & c \\ 0 & 1 & 2 \end{array}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 8 \end{vmatrix} = K(0-1)(1-2)(2-0)(0+1+2)$$

$6=6K \quad \therefore\ K=1$

$$\text{Hence } \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Note :—In case $a+b+c=0$, then clearly $\Delta=0$

Ex. 3. Prove that **(Pb. 45)**

$$\Delta = \begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$$

Putting $x=\alpha$ in Δ we find that column no. 1 & 4 become identical and hence $\Delta=0$

$\therefore$ $x-\alpha$ is a factor.

Again putting $x=\beta$ in Δ we get

$$\Delta=\begin{vmatrix} \beta & l & m & 1 \\ \alpha & \beta & n & 1 \\ \alpha & \beta & \beta & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix}$$

Subtract α and β times column four from column 1 and 2 respectively.

$$\Delta=\begin{vmatrix} \beta-\alpha & l-\beta & m & 1 \\ 0 & 0 & n & 1 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & \gamma & 1 \end{vmatrix} \text{ which is clearly zero}$$

$\therefore$ $x-\beta$ is a factor

Similarly we can show that $x-\gamma$ is also a factor

$$\Delta=K(x-\alpha)(x-\beta)(x-\gamma)$$

The value of K can be easily shown to be unity by comparing the coefficient of x^3 (the diagonal term).

*18. Multiplication of determinants.

(Agra 36, 45, Pb. 40, 43, 46)

Let $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta'=\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then $\Delta\ \Delta'=D$ where D is the determinant below.

$$\begin{matrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{matrix}$$

Now by §14 the above determinant can be expressed as the sum of 27 determinants. A few of which will be.

$$\begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 \end{vmatrix}, \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix}, \begin{vmatrix} a_1\alpha_1 & c_1\gamma_2 & b_1\beta_3 \\ a_2\alpha_1 & c_2\gamma_2 & b_2\beta_3 \\ a_3\alpha_1 & c_3\gamma_2 & b_3\beta_3 \end{vmatrix}$$

(1) (2) (3)

$$1 = \alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{vmatrix} = 0 \quad \because \text{ Identical lines}$$

$$2 = \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3\,\Delta$$

$$3 = \alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = -\alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

[On interchanging 2nd and 3rd column] $= -\alpha_1\gamma_2\beta_3\,\Delta$

Similarly it can be seen that out of 27 component determinants of D, 21 will vanish whereas the rest 6 can be written as

$$\alpha_1\beta_2\gamma_3\,\Delta,\ -\alpha_1\beta_3\gamma_2\,\Delta,\ -\alpha_2\beta_1\gamma_3\,\Delta,\ \alpha_2\beta_3\gamma_1\,\Delta$$
$$\alpha_3\beta_1\gamma_2\,\Delta, -\alpha_3\beta_2\gamma_1\,\Delta$$

Summing up these six determinants we get.

$$D = \Delta[\alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) - \alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + \alpha_3(\beta_1\gamma_2 - \beta_2\gamma_1)]$$

$$D = \Delta \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \Delta \times \Delta'$$

∴ In the same way we can multiply the determinant of any order.

By multiplying a row of Δ with a row of Δ', we mean to multiply the corresponding constituents of two rows and adding them i. e. multiplying 1st row of Δ with 3rd row of Δ'.

$$\begin{array}{ccc} a_1 & b_1 & c_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{array} \quad \begin{array}{c} \text{we get} \\ \text{or} \end{array} \quad \begin{array}{l} a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ \Sigma\, a_1\alpha_3 \end{array}$$

Hence our rule of multiplication is as follows :—

Take the first row of Δ and multiply it successively with 1st, 2nd and 3rd rows of Δ' as explained above. The three expressions thus obtained will be constituents of the 1st row (in order) of D.

Then take up second row af Δ and multiply it successively with 1st, 2nd and 3rd rows of Δ' as explained above. The three expressions thus obtained will be constituents of 2nd row (in order) of D.

In a similar manner 3rd row of D is obtained.

***Example 4. Find the value of**

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$$

(Agra 27, 47 Supp.)

In accordance with the rule in the previous article, the constituents of the first row of the new determinant will be

$$\text{1st constituent} = \lambda(a^2+\lambda^2)+c(ab+c\lambda)-b(ca-b\lambda)$$
$$=\lambda(a^2+b^2+c^2+\lambda^2)$$

$$\text{2nd constituent} = -c(a^2+\lambda^2)+\lambda(ab+c\lambda)+a(ca-b\lambda)$$
$$=0$$

$$\text{3rd constituent} = b(a^2+\lambda^2)-a(ab+c\lambda)+\lambda(ca-b\lambda)=0$$

Similarly find the constituents of 2nd and 3rd rows.

$$\therefore\ D=\begin{vmatrix} \lambda(a^2+b^2+c^2+\lambda^2) & 0 & 0 \\ 0 & \lambda(a^2+b^2+c^2+\lambda^2) & 0 \\ 0 & 0 & \lambda(a^2+b^2+c^2+\lambda^2) \end{vmatrix}$$

$=\lambda^3(a^2+b^2+c^2+\lambda^2)^3.$

Example 5. Express

$$\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$ as a determinant.

$$\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\times\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$\Delta=\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

§ 19. Solution of three equations of first degree in three unknown quantities. (Pb. 32, 36, 38, Agra 39)

Let the three equations be

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Now $\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$. On substituting the values of d_1, d_2 and d_3, we get

$$=\begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1y & b_1 & c_1 \\ b_2y & b_2 & c_2 \\ b_3y & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} c_1z & b_1 & c_1 \\ c_2z & b_2 & c_2 \\ c_3z & b_3 & c_3 \end{vmatrix}$$

$$= x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The last two determinants vanish.

$$\therefore \quad x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Similarly } y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{and } z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

§20. Notation.

We shall in the questions denote columns by C and rows by R. Thus C_1, C_2, C_3 stand for 1st, 2nd and 3rd columns respectively. Similarly R_1, R_2, R_3 stand for 1st 2nd and 3rd rows respectively.

Sometimes it is required while solving the determinants to alter rows and columns,

Suppose we are required to add twice the constituents of second row and subtract thrice the constituents of third row from the corresponding constituents of first row we shall express it as $R_1+2R_2-3R_3$.

Similarly $C_2-aC_3+bC_1$ would mean.

Subtract a times column no. 3 and add b times column no. 1 to column no. 2.

***Example 6 Evaluate** (Agra 50, 53)

$$\Delta=\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

If we expand here directly, the expansion will involve lengthy products and we are likely to commit mistakes. Hence efforts should always be made to reduce the numbers to lesser numbers and to bring in as many zeros as can be possible in any row or column and then expand with that row and column.

Apply $C_1+C_3-2C_2$ which in accordance with our notation above means that in the constituents of 1st column add the corresponding constituents of 3rd column and subtract twice the corresponding constituents of 2nd column. By the above application the first column of Δ will change but there will be no change in the value of Δ (Art 15).

$$\therefore\ \Delta=\begin{vmatrix} 265+219-2(240) & 240 & 219 \\ 240+198-2(225) & 225 & 198 \\ 219+181-2(198) & 198 & 181 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 240 & 219 \\ -12 & 225 & 198 \\ 4 & 198 & 181 \end{vmatrix}$$

Here again we can make two zeros in column no. 1 by applying R_2+3R_1 and R_3-R_1, thus changing R_2 and R_3 of Δ.

$$\therefore\ \Delta=\begin{vmatrix} 4 & 240 & 219 \\ -12+3(4) & 225+3(240) & 198+3(219) \\ 4-4 & 198-240 & 181-219 \end{vmatrix}$$

$$=\begin{vmatrix} 4 & 240 & 219 \\ 0 & 945 & 855 \\ 0 & -42 & -38 \end{vmatrix} \text{Now expand with 1s}$$

$$=4\begin{vmatrix} 945 & 855 \\ -42 & -38 \end{vmatrix}$$

Here again constituents of R_1 and R_2 has as common factor 45 and -2 resp. which can be taken out (Art. 13)

$$\therefore\ \Delta=4(45)(-2)\begin{vmatrix} 21 & 19 \\ 21 & 19 \end{vmatrix}=0$$

Because C_1 and C_2 are identical (Art. 10).

EXERCISE No. 9

Note :—Before doing the questions, read Art. 20 and Ex. 5 very throughly.

1. Evaluate the following determinants.

(a) $$\Delta=\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix} \quad \text{Now } \Delta=\begin{vmatrix} 7.6+1 & 1 & 6 \\ 7.4+7 & 7 & 4 \\ 7.2+3 & 3 & 2 \end{vmatrix}$$

Break into sum of two determinants each of which will be zero because of identical lines $\therefore\ \Delta=0$

(b) $$\Delta=\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$ **Where ω is a cube root of unity**

We know that if ω is a cube root of unity then

$1+\omega+\omega^2=0$ and $\omega^3=1$ and $\omega^4=\omega^3.\omega=\omega$

Apply $c_1+c_2+c_3$ and thus every constituent of c_1 becomes

$1+\omega+\omega^2=0 \quad \therefore \Delta=0$

(c) $\Delta=\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ **(Nagpur 33)**

Proceeding as in part (b) $\Delta=0$

*(d) $\Delta=\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

K 123

Apply c_3+c_2 and take out $a+b+c$ and thus c_1 and c_3 of the new determinant become identical $\therefore \Delta=0$

(Pb. 35)

(e) $\Delta=\begin{vmatrix} a+2b & a+4b & a+6b \\ a+3b & a+5b & a+7b \\ a+4b & a+6b & a+8b \end{vmatrix}$

Apply R_2-R_1 and R_3-R_1 and thus R_2 and R_3 of new new determinant become identical $\therefore \Delta=0$

*(f) **Evaluate** $\Delta=\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

$$\Delta=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}-\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \quad \text{(Art. 15)}$$

$$=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}-\frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} \quad \text{(Art. 13 Note I)}$$

Now take abc common from c_3 of 2nd determinant and make c_3 pass over two columns and thus becoming column No. 1. Hence 2nd determinant becomes same as 1st.

$\therefore \quad \Delta=0.$

(g) **Evaluate** $\Delta=\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$. Put $w^3=1$ and apply c_1-c_2 and then expand $\therefore \Delta=3$. ?

(h) **Evaluate** $\Delta=\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$ **(Pb. 40, Alld. 47)**

Apply C_1-3C_3, C_2-2C_3, C_4-4C_3 and expand thus getting a determinant of 3rd order in which again apply R_3-R_1 and R_2-R_1 and exyand with 3rd row. $\therefore \quad \Delta=6$

*(i) **Evaluate** $\Delta=\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$ **(Benares 49)**

Apply R_1-7R_4, R_2-6R_4, R_3-2R_4 etc. $\therefore \quad \Delta=0.$

(j) **Evaluate** $\Delta=\begin{vmatrix} 3 & 7 & 9 & 6 \\ 8 & 4 & 5 & 8 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$ **(Agra 56)**

Applying $R_2 - R_4$ we get

$$\Delta = \begin{vmatrix} 3 & 7 & 9 & 6 \\ 2 & 2 & -4 & 0 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$$

Again applying $C_2 - C_1$, $C_3 + 2C_1$, we get

$$\Delta = \begin{vmatrix} 3 & 4 & 15 & 6 \\ 2 & 0 & 0 & 0 \\ 7 & 3 & 17 & 5 \\ 6 & -4 & 21 & 8 \end{vmatrix}$$

Expanding with the help of 2nd row

$$\Delta = -2 \begin{vmatrix} 4 & 15 & 6 \\ 3 & 17 & 5 \\ -4 & 21 & 8 \end{vmatrix}$$

Adding R_1 to R_3, we get

$$\Delta = -2 \begin{vmatrix} 4 & 15 & 6 \\ 3 & 17 & 5 \\ 0 & 36 & 14 \end{vmatrix}$$

$$= -2[4(238-180) - 3(210-216)]$$

$$= -2[232+18] = -500.$$

***(k) Evaluate**

$$\Delta = \begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$$

(Agra 32, Patna 49, Rajputana 53)

By multiplying a row of Δ with a row of Δ', we mean to multiply the corresponding constituents of two rows and adding them i. e. multiplying 1st row of Δ with 3rd row of Δ'.

$$\begin{matrix} a_1 & b_1 & c_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{matrix} \quad \begin{matrix} \text{we get} & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ \text{or} & \Sigma\, a_1\alpha_3 \end{matrix}$$

Hence our rule of multiplication is as follows :—

Take the first row of Δ and multiply it successively with 1st, 2nd and 3rd rows of Δ' as explained above. The three expressions thus obtained will be constituents of the 1st row (in order) of D.

Then take up second row af Δ and multiply it successively with 1st, 2nd and 3rd rows of Δ' as explained above. The three expressions thus obtained will be constituents of 2nd row (in order) of D.

In a similar manner 3rd row of D is obtained.

***Example 4. Find the value of**

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}.$$

(Agra 27, 47 Supp.)

In accordance with the rule in the previous article, the constituents of the first row of the new determinant will be

$$\text{1st constituent} = \lambda(a^2+\lambda^2)+c(ab+c\lambda)-b(ca-b\lambda)$$
$$=\lambda(a^2+b^2+c^2+\lambda^2)$$

$$\text{2nd constituent} = -c(a^2+\lambda^2)+\lambda(ab+c\lambda)+a(ca-b\lambda)$$
$$=0$$

$$\text{3rd constituent} = b(a^2+\lambda^2)-a(ab+c\lambda)+\lambda(ca-b\lambda)=0$$

Similarly find the constituents of 2nd and 3rd rows.

$$\therefore D=\begin{vmatrix} \lambda(a^2+b^2+c^2+\lambda^2) & 0 & 0 \\ 0 & \lambda(a^2+b^2+c^2+\lambda^2) & 0 \\ 0 & 0 & \lambda(a^2+b^2+c^2+\lambda^2) \end{vmatrix}$$

$=\lambda^3(a^2+b^2+c^2+\lambda^2)^3.$

Example 5. Express

$$\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 \text{ as a determinant.}$$

$$\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$\Delta=\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

§ 19. **Solution of three equations of first degree in three unknown quantities.** (Pb. 32, 36, 38, Agra 39)

Let the three equations be

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Now $\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$. On substituting the values of d_1, d_2 and d_3, we get

$$=\begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1y & b_1 & c_1 \\ b_2y & b_2 & c_2 \\ b_3y & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} c_1z & b_1 & c_1 \\ c_2z & b_2 & c_2 \\ c_3z & b_3 & c_3 \end{vmatrix}$$

$$= x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

$\because$ The last two determinants vanish.

$$\therefore \quad x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Similarly } y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{and } z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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Subtract a times column no. 3 and add *b* times column no. 1 to column no. 2.

***Example 6** **Evaluate** **(Agra 50, 53)**

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$$=\begin{vmatrix} 4 & 240 & 219 \\ 0 & 945 & 855 \\ 0 & -42 & -38 \end{vmatrix} \text{Now expand with 1st column}$$

$$=4\begin{vmatrix} 945 & 855 \\ -42 & -38 \end{vmatrix}$$

Here again constituents of R_1 and R_2 has as common factor 45 and -2 resp. which can be taken out (Art. 13)

$$\therefore\ \Delta=4(45)(-2)\begin{vmatrix} 21 & 19 \\ 21 & 19 \end{vmatrix}=0$$

Because C_1 and C_2 are identical (Art. 10).

EXERCISE No. 9

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Break into sum of two determinants each of which will be zero because of identical lines $\therefore\ \Delta=0$

(b) $$\Delta=\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$ **Where ω is a cube root of unity**

We know that if ω is a cube root of unity then

$1+\omega+\omega^2=0$ and $\omega^3=1$ and $\omega^4=\omega^3.\omega=\omega$

Apply $c_1+c_2+c_3$ and thus every constituent of c_1 becomes $1+\omega+\omega^2=0 \quad \therefore \Delta=0$

(c) $$\Delta=\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$ (Nagpur 33)

Proceeding as in part (b) $\Delta=0$

*(d) $$\Delta=\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$ Apply c_3+c_2 and take out $a+b+c$ and thus c_1 and c_3 of the new determinant become identical $\therefore \Delta=0$ (Pb. 35)

(e) $$\Delta=\begin{vmatrix} a+2b & a+4b & a+6b \\ a+3b & a+5b & a+7b \\ a+4b & a+6b & a+8b \end{vmatrix}$$ Apply R_2-R_1 and R_3-R_1 and thus R_2 and R_3 of new new determinant become identical $\therefore \Delta=0$

*(f) Evaluate $$\Delta=\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$$

$$\Delta=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$ (Art. 15)

$$=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$ (Art. 13 Note I)

Now take abc common from c_3 of 2nd determinant and make c_3 pass over two columns and thus becoming column No. 1. Hence 2nd determinant becomes same as 1st.

$\therefore \quad \Delta=0.$

(g) **Evaluate** $\Delta=\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$. Put $\omega^3=1$ and apply c_1-c_2 and then expand $\therefore$ $\Delta=3$.

(h) **Evaluate** $\Delta=\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$ **(Pb. 40, Alld. 47)**

Apply C_1-3C_3, C_2-2C_3, C_4-4C_3 and expand thus getting a determinant of 3rd order in which again apply R_3-R_1 and R_2-R_1 and exyand with 3rd row. $\therefore \quad \Delta=6$

*(i) **Evaluate** $\Delta=\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$ **(Benares 49)**

Apply R_1-7R_4, R_2-6R_4, R_3-2R_4 etc. $\therefore$ $\Delta=0$

(j) **Evaluate** $\Delta=\begin{vmatrix} 3 & 7 & 9 & 6 \\ 8 & 4 & 5 & 8 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$ **(Agra 56)**

Applying $R_2 - R_4$ we get

$$\Delta = \begin{vmatrix} 3 & 7 & 9 & 6 \\ 2 & 2 & -4 & 0 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$$

Again applying $C_2 - C_1$, $C_3 + 2C_1$, we get

$$\Delta = \begin{vmatrix} 3 & 4 & 15 & 6 \\ 2 & 0 & 0 & 0 \\ 7 & 3 & 17 & 5 \\ 6 & -4 & 21 & 8 \end{vmatrix}$$

Expanding with the help of 2nd row

$$\Delta = -2\begin{vmatrix} 4 & 15 & 6 \\ 3 & 17 & 5 \\ -4 & 21 & 8 \end{vmatrix}$$

Adding R_1 to R_3, we get

$$\Delta = -2\begin{vmatrix} 4 & 15 & 6 \\ 3 & 17 & 5 \\ 0 & 36 & 14 \end{vmatrix}$$

$$= -2[4(238-180)-3(210-216)]$$

$$= -2[232+18] = -500.$$

*(k) Evaluate

$$\Delta = \begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$$

(Agra 32, Patna 49, Rajputana 53)

$$\Delta=\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}+\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}=D+D_1$$

Multiply R_1, R_2, R_3 and R_4 of D_1 by a, b, c and d respectively and hence divide the determinant by $abcd$.

$$\therefore\quad D_1=\frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}=\begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$$

$$=(-1)^3\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}=-D \quad\therefore\quad D+D_1=0$$

Hence $\Delta=0$.

(1) (i) If ω is a cube-root of unity, prove that $a+b\omega+c\omega^2$ is a factor of

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

(Agra 54)

Hence evaluate the determinant.

(ii) Find the value of

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

(Dacca 46, Mysore 49, Agra 54)

(i) If ω be a cube root of unity then $\omega^3=1$ and $\omega+w^2=-1$ and $\omega^4=\omega^3\,.\,\omega=\omega$

Apply $R_1+R_2+R_3$ and the elements of Ist row in the transformed determinant are $a+b+c$, $a+b+c$, $a+b+c$ showing that $a+b+c$ is a factor. Again apply $R_1+\omega R_2+\omega^2 R_3$ we get 1st row as

$$a+b\omega+c\omega^2,\ b+c\omega+a\omega^2,\ c+a\omega+b\omega^2,$$

$$\text{or } a+b\omega+c\omega^2,\ \frac{1}{\omega}(b\omega+c\omega^2+a\omega^3),\ \frac{1}{\omega^2}(c\omega^2+a\omega^3+b\omega^4)$$

$$\text{or } a+b\omega+c\omega^2,\ \frac{1}{\omega}(a+b\omega+c\omega^2),\ \frac{1}{\omega^2}(a+b\omega+c\omega^2)$$

showing that $a+b\omega+c\omega^2$ is a factor.

Similarly we can show that $a+b\omega^2+c\omega$ is also a factor

$\therefore\ K(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)=\Delta$

Comparing coefficient of a^3 we get $K=-1$

$\therefore\ -1(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=\Delta$

or $-(a^3+b^3+c^3-3abc)=\Delta$

(ii) $\Delta=\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$ Apply c_2-4c_1, c_3-9c_1, c_4-16c_1 and expand.

$$\therefore\ \Delta=1\begin{vmatrix} -7 & -20 & -39 \\ -20 & -56 & -108 \\ -39 & -108 & -207 \end{vmatrix}$$

Now we have to expand but in order to make smaller digits we apply c_3-c_2 and c_2-c_1 after taking minus from each column.

$$\Delta=-\begin{vmatrix} 7 & 13 & 19 \\ 20 & 36 & 52 \\ 39 & 69 & 99 \end{vmatrix} \quad \text{Again apply } R_2-3R_1$$

$$\Delta=-\begin{vmatrix} 7 & 13 & 19 \\ -1 & -3 & -5 \\ 39 & 69 & 99 \end{vmatrix}=\begin{vmatrix} 7 & 13 & 19 \\ 1 & 3 & 5 \\ 39 & 69 & 99 \end{vmatrix}$$

Again apply c_2-3c_1 and c_3-5c_1 we get

$$\Delta = \begin{vmatrix} 7 & -8 & -16 \\ 1 & 0 & 0 \\ 39 & -48 & -96 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 7 & 8 & 16 \\ 39 & 48 & 96 \end{vmatrix}$$

$$= -1(8 \times 96 - 16 \times 48) = 0$$

Q. 2. Prove the following :—

*(a) $$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

(Agra 29, Luck. 45)

To prove this you have to change rows into columns and make some more movements of lines (Art 6, 7)

(b) $$\begin{vmatrix} a & b & c & 0 \\ b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{vmatrix} = \begin{vmatrix} -a & b & c & 0 \\ b & -a & 0 & c \\ c & 0 & -a & b \\ 0 & c & b & -a \end{vmatrix}$$

Multiply the first and fourth column by (-1) and divide the determinant by $(-1)^2$ i. e. by 1

$$\therefore \Delta = \begin{vmatrix} -a & b & c & 0 \\ -b & a & 0 & -c \\ -c & 0 & a & -b \\ 0 & c & b & -a \end{vmatrix}$$

Now take (-1) common from R_2 and R_3.

(c) $$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Apply on L. H. S. $c_1 - c_2 + c_3$ and take 2 common

$$\therefore \text{L. H. S.} = 2 \begin{vmatrix} b & c+a & a+b \\ q & r+p & p+q \\ y & z+x & x+y \end{vmatrix}$$

Now break into four determinants all of which shall vanish except one which shall be easily shown to be R. H. S.

*(d)

$$\Delta=\begin{vmatrix} \beta\gamma\delta & \alpha & \alpha^2 & \alpha^3 \\ \gamma\delta\alpha & \beta & \beta^2 & \beta^3 \\ \delta\alpha\beta & \gamma & \gamma^2 & \gamma^3 \\ \alpha\beta\gamma & \delta & \delta^2 & \delta^3 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}$$

Proceed as for D_1 of part (k) in Q. 1 P. 264.

*(e) If $u=ax^2+2bxy+cy^2$ and $u'=a'x^2+2b'xy+c'y^2$.

Prove that

$$\Delta=\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

$$= -\frac{1}{y}\begin{vmatrix} u & u' \\ ax+by & a'x+b'y \end{vmatrix}$$

Multiply c_1 and c_3 by x and y resp. and hence divide Δ by xy.

$$\therefore\ \Delta=\frac{1}{xy}\begin{vmatrix} xy^2 & -xy & x^2y \\ ax & b & cy \\ a'x & b' & c'y \end{vmatrix}$$ Apply c_1+yc_2 and c_3+xc_2

$$=\frac{1}{xy}\begin{vmatrix} 0 & -xy & 0 \\ ax+by & b & bx+cy \\ a'x+b'y & b' & b'x+c'y \end{vmatrix}$$

$$=\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

$$=\frac{1}{y}\begin{vmatrix} ax+by & bxy+cy^2 \\ a'x+b'y & b'xy+c'y^2 \end{vmatrix}$$ Now apply C_2+xC_1

$$=\frac{1}{y}\begin{vmatrix} ax+by & ax^2+2bxy+cy^2 \\ a'x+b'y & a'x^2+2b'xy+c'y^2 \end{vmatrix}$$

$$=\frac{1}{y}\begin{vmatrix} ax+by & u \\ a'x+b'y & u' \end{vmatrix}=\frac{1}{y}\begin{vmatrix} ax+by & a'x+b'y \\ u & u' \end{vmatrix}$$

$$=-\frac{1}{y}\begin{vmatrix} u & u' \\ ax+by & a'x+b'y \end{vmatrix}$$ We have changed two adjacent rows and hence $-$ive sign.

*(i) $\Delta=\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}=-(ac-b^2)(ax^2+2bxy+cy^2)$ (Sagar 49)

Apply $R_3-(xR_1+yR_2)$ and expand with 3rd row of new determinant thus obtained

(g) If $u=ax^4+4bx^3+6cx^2+4dx+e$

$u_{11}=ax^2+2bx+c$

$u_{12}=bx^2+2cx+d$

$u_{22}=cx^2+2dx+e$

Prove $\Delta=\begin{vmatrix} a & b & c & u_{11} \\ b & c & d & u_{12} \\ c & d & e & u_{22} \\ u_{11} & u_{12} & u_{22} & 0 \end{vmatrix}=-u\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$

Substitute the values of u_{11}, u_{12}, and u_{22} in Δ

Apply $R_4-(x^2R_1+2xR_2+R_3)$ thus getting three zeros in 4th row of the new determinant thus obtained.

(h) **Prove that** $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$ **is a perfect square.**

Agra 41)

Take out a, b and c as common factors from c_1, c_2 and c_3 respectively and then in the new determinant thus obtained make two zeros. $\Delta = 4a^2b^2c^2$.

(i) **Prove** $\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$ Proceed as in (h) part.

(j) **Prove** $\Delta = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$

Applying $c_1+c_2+c_3$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

$\because$ 0 can be written as $0(a+b+c)$.

Apply R_3-2R_1 and expand with column No. 1.

(k) **Prove** $\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{c^2+a^2}{b} \end{vmatrix} = 4abc$

Multiply R_1, R_2 and R_3 by c, a and b respectively and hence divide Δ by abc.

$$\therefore \quad \Delta=\frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

Now make one zero in column no. 2 by c_2-c_3 and then expand.

*(l) **Prove**

$$\Delta=\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}=(a+b+c)^3$$

(Rangoon 50, Allahabad 37)

Apply C_1-C_2, C_2-C_3.

$$\Delta=(a+b+c)^2\begin{vmatrix} -1 & 0 & 2a \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Now apply R_2+R_1 and expand.

*(m) **Prove**

$$\begin{vmatrix} a+b+nc & (n-1)a & (n-1)b \\ (n-1)c & b+c+na & (n-1)b \\ (n-1)c & (n-1)a & c+a+nb \end{vmatrix}=n(a+b+c)^3$$

Apply $C_1+C_2+C_3$ and take out $n(a+b+c)$ common from c_1 of the new determinant and then make two zeros in the c_1 and expand.

*(n) **Prove**

$$\begin{vmatrix} 1+a_1 & a_2 & a_3 & a_4 \\ a_1 & 1+a_2 & a_3 & a_4 \\ a_1 & a_2 & 1+a_3 & a_4 \\ a_1 & a_2 & a_3 & 1+a_4 \end{vmatrix}$$

$$=1+a_1+a_2+a_3+a_4$$

(Agra 1953)

Apply $c_1+c_2+c_3+c_4$ and take out $1+a_1+a_2+a_3+a_4$ common from c_1 of the new determinant and then make three zeros in c_1 by applying R_2-R_1, R_3-R_1 and R_4-R_1

$$(n_1)\begin{vmatrix} 1+a_1 & a_2 & a_3 \ldots\ldots a_n \\ a_1 & 1+a_2 & a_3 \ldots\ldots a_n \\ \vdots & \vdots & \vdots \qquad \vdots \\ a_1 & a_2 & a_3 \ldots\ldots 1+a_n \end{vmatrix} = 1+a_1+a_2+\ldots\ldots a_n$$

(Agra 38)

*(o) Prove
$$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} = x^3(x+10)$$

(Agra 39, Patna 41, Mysore 42, Luck. 49)

$$(o_1)\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)^3$$

(Agra 43, Luck. 43 Supp. Delhi 47, Nag. 36)

Procced exactly as in part (n).

*(p) Prove
$$\begin{vmatrix} x^3 & 3x^2 & 3x & 1 \\ x^2 & x^2+2x & 2x+1 & 1 \\ x & 2x+1 & x+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (x-1)^6$$

(Agra 41, Luck. 49)

Apply $C_1-C_2+C_3-C_4$ we get

$$\Delta=(x-a)^3\begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Now make two zeros in c_3 and expand by it.

*(q) Prove $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix}=(ax-by+cz)^2$

(Rajputana 52)

$$\Delta=\frac{1}{a}\begin{vmatrix} 0 & ax & y & z \\ -x & 0 & c & b \\ -y & -ac & 0 & a \\ -z & -ab & -a & 0 \end{vmatrix}$$

Apply $C_2-bC_3+cC_4$

$$\Delta=\frac{1}{a}\begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0 & c & b \\ -y & 0 & 0 & a \\ -z & 0 & -a & 0 \end{vmatrix}$$

Expanding with C_2 and hence $-$ive sign.

$$\therefore\ \Delta=-\frac{ax-by+cz}{a}\begin{vmatrix} -x & c & b \\ -y & 0 & a \\ -z & -a & 0 \end{vmatrix}$$

$$=\frac{ax-by+cz}{a}\begin{vmatrix} x & c & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}=\frac{ax-by+cz}{a^2}\begin{vmatrix} ax & ac & ab \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

Apply $R_1-bR_2+cR_3$ etc.

(r) **Prove** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix} = abc$

Make three zeros in c_1 and expand.

(s) **Prove** $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

(I. A. A. S. 44, Calcutta 48, Pb. 48, Agra 53)

Divide R_1, R_2, R_3 and R_4 by a, b, c and d respectively.

$$\therefore\ \Delta = abcd \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}$$

Now apply $R_1+R_2+R_3+R_4$ and proceed as in part (n).

*(t) **Prove**,

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac & ad \\ ab & b^2+1 & bc & bd \\ ac & bc & c^2+1 & cd \\ ad & bd & cd & d^2+1 \end{vmatrix} = \begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

$= 1+a^2+b^2+c^2+d^2$ (Allahabad 49)

Multiply c_1. c_2, c_3 and c_4 by a, b, c and d respectively and hence dividing by $abcd$,

$$\Delta=\frac{1}{abcd}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 & ad^2 \\ a^2b & b(b^2+1) & bc^2 & bd^2 \\ a^2c & b^2c & c(c^2+1) & cd^2 \\ a^2d & b^2d & cd^2 & d(d^2+1) \end{vmatrix}$$

Now a, b, c and d can be taken out from R_1, R_2, R_3 and R_4 respectively.

$$\therefore \quad \Delta=\frac{abcd}{abcd}\begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

Now proceed as in part n.

*(u) Show that

$$\Delta=\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

is divisible by λ^3 and find the other factor. (Rajputana 49)

Proceeding exactly as in (t) part above, we find

$$\Delta=(\lambda+a^2+b^2+c^2+d^2)\lambda^3.$$

Hence it is divisible by λ^3 and the other factor is

$$\lambda+a^2+b^2+c^2+d^2$$

(v) $$\Delta=\begin{vmatrix} 1 & a & a^2 & 0 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{vmatrix}=1+a^4+a^8$$

(Dacca 43)

Apply $c_1+c_2+c_3+c_4$ and take out the common factor $1+a+a^2$.

$$\Delta=1+a+a^2\begin{vmatrix} 1 & a & a^2 & 0 \\ 1 & 1 & a & a^2 \\ 1 & 0 & 1 & a \\ 1 & a^2 & 0 & 1 \end{vmatrix}$$

Apply R_2-R_1, R_3-R_1 and R_4-R_1.

$$\Delta=1+a+a^2\begin{vmatrix} 1-a & a-a^2 & a^2 \\ -a & 1-a^2 & a \\ c^2-a & -a^2 & 1 \end{vmatrix}$$

Apply $R_1-R_2+R_3$

$$=1+a+a^2\begin{vmatrix} 1+a^2-a & -(1+a^2-a) & 1+a^2-a \\ -a & 1-a^2 & a \\ a^2-a & -a^2 & 1 \end{vmatrix}.$$

Take out $1+a^2-a$ common from R_1.

$$=[(1+a^2)^2-a^2]\begin{vmatrix} 1 & -1 & 1 \\ -a & 1-a^2 & a \\ a^2-a & -a^2 & 1 \end{vmatrix}$$

Now apply c_2+c_1 and c_3-c_1 and then expand.

(v) Prove

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}=(1+a^2+b^2)^3$$

(Pb. 34)

Apply C_1-bC_3 and C_2+aC_3 and take out $(1+a^2+b^2)$ each common from c_1 and c_2 of the new determinant thus obtained.

$$\therefore\quad \Delta=(1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$ Now expand.

*(x) **Prove**

$$\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix}=abc(a^2+b^2+c^2)^3$$

Take a, b, c common from c_1, c_2 and c_3 respectively.

$$=abc\begin{vmatrix} a^2-b^2-c^2 & 2b^2 & 2c^2 \\ 2a^2 & b^2-c^2-a^2 & 2c^2 \\ 2a^2 & 2b^2 & c^2-a^2-b^2 \end{vmatrix}$$

Apply R_2-R_1 and R_3-R_1 and take $(a^2+b^2+c^2)$ each common from R_2 and R_3 of the new determinant thus obtained and then expand.

(y) **Prove** $$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}=(a^2+b^2+c^2+d^2)^2$$

$$\Delta=\frac{1}{a}\begin{vmatrix} a^2 & b & c & d \\ -ab & a & -d & c \\ -ac & d & a & -b \\ -ad & -c & b & a \end{vmatrix}$$

Apply $C_1+bC_2+cC_3+dC_4$

$$=\frac{1}{a}\begin{vmatrix} a^2+b^2+c^2+d^2 & b & c & d \\ 0 & a & -d & c \\ 0 & d & a & -b \\ 0 & -c & b & a \end{vmatrix}$$

$$=\frac{a^2+b^2+c^2+d^2}{a}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$ Now expand.

***(y) Prove** $\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}=a^4$

Apply R_2-R_1, R_3-R_1, R_4-R_1

$$\Delta=a^2\begin{vmatrix} 1 & a+b & a+b+c \\ 2 & 3a+2b & 4a+3b+2c \\ 3 & 6a+3b & 10a+6b+3c \end{vmatrix}$$

Again apply R_2-2R_1, and R_3-3R_1 and then expand.

***(z) Find the value of**

$$\Delta=\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}$$

(Agra 43)

Apply, C_5-C_4

$$\Delta=\begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 1 \\ 1 & 3 & 6 & 10 & 5 \\ 1 & 4 & 10 & 20 & 15 \\ 1 & 5 & 15 & 35 & 34 \end{vmatrix}$$

Aagain apply in succession C_4-C_3, C_3-C_2 and C_2-C_1

$$\Delta=\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 34 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 34 \end{vmatrix}$$

Again apply in succession C_4-C_3, C_3-C_2 and C_2-C_1

$$\Delta=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 14 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 10 & 14 \end{vmatrix}$$

Again apply in succession C_3-C_2 and C_2-C_1

$$\Delta=\begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 6 & 4 & 4 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ 4 & 4 \end{vmatrix}=4-4=0$$

Q. 3. Prove the following identities.

***(a)**
$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}=(x-y)(y-z((z-x)(xy+yz+zx)$$

(Delhi 40, Calcutta 46, Pb. 48, Luck. 49, Supp., Utkal 50)

Multiply c_1, c_2 and c_3 by x, y and z respectively and hence divide the determinant by xyz.

$$\therefore \quad \Delta = \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (-1)^2\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} \text{(Art. 8)} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

Put $x=y$, we find $c_1=c_2$. $\therefore$ determinant vanishes by art. 10 and hence $x-y$ is a factor.

Similarly $y-z$ and $z-x$ are also the factors of above.

$\therefore$ $(x-y)(y-z)(z-x)$ is a third degree factor of the determinant which is of fifth degree as is judged from diagonal term y^2z^3. $\therefore$ The remaining factor must be of 2nd degree. But the determinant is symmetrical and also the three factors found are symmetrical.

$\therefore$ The remaining 2nd degree factor must also be symmetrical. Let it be $[L(x^2+y^2+z^2)+M(xy+yz+zx)]$

$$\therefore \quad \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)[L(x^2+y^2+z^2) + M(xy+yz+zx)]$$

Putting $x=0, y=1$ and $z=-1$ in both sides and simplifying we get $-2=2(2L-M)$ $\therefore$ $2L-M=-1$.........(A)

Again putting $x=0, y=1, z=2$, a different set of values, we get $4=2(5L+2M)$ $\therefore$ $5L+2M=2$.......(B)

Solving (A) and (B) we find $L=0$ and $M=1$.

$\therefore$ Substituting the values of L and M, factors are found to be $(x-y)(y-z)(z-x)(xy+yz+zx)$. (Pb. 1932) *(a₁) $$\Delta=\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}=-(a+b+c)(a-b)(b-c)(c-a)$$

(Patna 23)

Apply c_2+c_1, then $$\Delta=-(a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ etc. from Ex. 1 P. 252

(b) Simplify $$\begin{vmatrix} a^3 & a & 1 \\ b^3 & b & 1 \\ c^3 & c & 1 \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$ or $D \div D_1$

Now by Ex. 1 and 2 of page 252 factors of D and D_1 are $(a\ \ b)(b-c)(c-a)(a+b+c)$ and $(a-b)(b-c)(c-a)$ respectively. $\therefore$ $D \div D_1=a+b+c$.

(c) Factorize

$$\Delta=\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$=abc(a-b)(b-c)(c-a)$ by Ex. 1 Art. 17 Page 252

*(d) **Prove that**

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & a^3 \end{vmatrix}=(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

(Pb. 39, 44)

Putting $a=b$ and so on we find the following factors .—

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

of sixth degree of the given determinant which too is of the same degree. Hence if there can be any other factor that must be a constant say K,

$$\therefore \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = K(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

Comparing the coefficient of diagonal term

bc^2d^3 we find $K=1$.

(e) If x, y and z are all different and given that **(Nagpur 30, Annamalai 38)**

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0 \quad \text{Prove } 1+xyz=0$$

$$\Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Hence by Ex. 1 art. 17 Page 252

$\Delta = (x-y)(y-z)(z-x)\{1+xyz\} = 0$

But because x, y and z are all different

$\therefore$ Neither of $x-y, y-z$ and $z-x$ is zero

Hence $1+xyz=0$

(f) If a, b and c are all different and given that

$$\Delta=\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix}=0 \text{ then } abc(bc+ca+ab)=a+b+c$$

Proceed as above and use Q. 3 (a) P. 278 and Ex. 2 Art. 17 P. 252.

Prove that

*(g)

$$\Delta=\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix}=(a-b)(a-c)(a-d)(b-c))b-d)(c-d)$$

(Agra 29, Pb. 51)

Put $bc+ad=\text{A}$, $ca+bd=\text{B}$ and $ab+cd=\text{C}$.

$$\therefore\ \Delta=\begin{vmatrix} 1 & \text{A} & \text{A}^2-2abcd \\ 1 & \text{B} & \text{B}^2-2abcd \\ 1 & \text{C} & \text{C}^2-2abcd \end{vmatrix}$$

Split into two determinants one of which will be zero because of identical lines and the other by Ex. 1 Art. 17 P. 252 will be $(\text{A}-\text{B})(\text{B}-\text{C})(\text{C}-\text{A})$. Now $\text{A}-\text{B}=bc+ad-ca-bd$ $=(c-d)(b-a)$ and so on.

Prove that

***(h)**

$$\Delta=\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}=(a+b+c+d)(a+d-b-c)(a+c-b-d)(a+b-c-d)$$

Punjab 42, 46, Nagpur 50)

Applying $c_1+c_2+c_3+c_4$, the new column number one shall give $a+b+c+d$ as a factor of Δ.

Similarly applying $c_1+c_2-c_3-c_4$

we find that $a+b-c-d$ will also be a factor

Again applying $c_1+c_3-c_2-c_4$ we get

$a+c-b-d$ as a factor

and by applying $c_1+c_4-c_2-c_3$ we get

$a+d-b-c$ as a factor

$\therefore$ $(a+b+c+d)(a+b-c-d)(a+c-b-d)(a+d-b-c)$ is a factor of fourth degree of $\triangle$ which is also of the same degree. Hence any other factor may be some constant K whose value is found to be unity by comparing the coefficient of a^4 the diagonal term.

Prove that

*(i)

$$\triangle=\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}=2abc(a+b+c)^3$$

(Calcutta 31, Agra 38, 42, 50, Patna 38, Pb. 41, Mysore 44, Annamalai 47)

Putting $a=0$ we find c_2 and c_3 become identical

$\therefore$ a is a factor.

Similarly b and c are the factors $\therefore$ abc is a factor of $\triangle$

Applying c_1-c_2 and c_2-c_3

$$\triangle=\begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ b^2-(c+a)^2 & (c+a)^2-b^2 & b^2 \\ 0 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$\triangle=\begin{vmatrix} (b+c+a)(b+c-a) & (b+c+a)0 & a^2 \\ (b+c+a)(b-c-a) & (b+c+a)(c+a-b) & b^2 \\ (b+c+a)0 & (b+c+a)(c-a-b) & (a+b)^2 \end{vmatrix}$$

Now c_1 and c_2 each give $(a+b+c)$ a factor

$\therefore$ $(a+b+c)^2$ is a factor of fifth degree of $\triangle$ which is of sixth degree. Now $\triangle$ being symmetrical and the factor found are also symmetrical hence the remaining factor must also be a symmetrical one and of first degree. Hence it must be $(a+b+c)$

$\therefore$ $abc(a+b+c)^3$ is a factor.

Let the constant factor if any be K then

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = K\,abc(a+b+c)^3 \ldots\ldots A$$

Putting $a=1$, $b=-1$, $c=1$ in both sides of A we find that $K=2$. Hence proved.

Note:—We have not put a or b or c equal to zero because in that case R. H. S. of A shall be zero.

*(j) Prove that

$$\Delta = \begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

Multiply R_1, R_2 and R_3 by c, a and b respectively and hence dividing the determinant by abc

$$\Delta = \frac{1}{abc}\begin{vmatrix} c(a+b)^2 & c^2a & c^2b \\ a^2c & a(b+c)^2 & a^2b \\ b^2c & b^2a & b(c+a)^2 \end{vmatrix}$$

Take c, a and b common from C_1, C_2 and C_3 respectively.

$$= \frac{abc}{abc}\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

which is of the same form as before in part (i).

*(k) Prove that (Agra 45)

$$\Delta = \begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix} = (x-y)(y-z)(\text{[illegible]}-x)(x+y+z)(x^2+y^2+z^2)$$

Putting $x=y$, we find that R_1 and R_2 become identical.
$\therefore$ $x-y$ is factor.

Similarly $(y-z)$ and $(z-x)$ are also factors.

Again if we change column no. 2 by applying C_2-4C_3 it becomes

$$\begin{matrix} x^2-[(y-z)^2+4yz] & & x^2-(y+z)^2 & & (x+y+z)(x-y-z) \\ y^2-[(z-x)^2+4zx] & \text{or} & y^2-(z+x)^2 & \text{or} & (x+y+z)(y \quad z-x) \\ z^2-[(x-y)^2+4xy] & & z^2-(x+y)^2 & & (x+y+z)(z-x-y) \end{matrix}$$

which shows that $x+y+z$ is a factor of c_3 and hence of Δ.

Again if we change column No. 2 of the original determinant by applying $c_2-2c_1-2c_3$, we find that it becomes

$$\begin{matrix} -(x^2+y^2+z^2) \\ -(x^2+y^2+z^2) \\ -(x^2+y^2+z^2) \end{matrix}$$

which shows that $x^2+y^2+z^2$ is a factor of c_2 and hence of Δ.

$\therefore$ $(x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$ are the sixth degree factors of Δ which too is of the same degree. Hence the remaining factor if there be any, let it be k which can be easily shown as usual to be unity.

Alternative method.

Split the given determinant into two, one of which shall vanish because of identical lines, we get

$$\Delta=\begin{vmatrix} x^2 & -(y^2+z^2)+2yz & yz \\ y^2 & -(z^2+x^2)+2zx & zx \\ z^2 & -(x^2+y^2)+2xy & xy \end{vmatrix}$$

Again split into two determinants one of which shall vanish $\therefore$ $\Delta=-\begin{vmatrix} x^2 & y^2+z^2 & yz \\ y^2 & z^2+x^2 & zx \\ z^2 & x^2+y^2 & xy \end{vmatrix}$

Applying c_1+c_2 and take out common factor $x^2+y^2+z^2$.

$$\Delta=-(x^2+y^2+z^2)\begin{vmatrix}1 & y^2+z^2 & yz\\ 1 & z^2+x^2 & zx\\ 1 & x^2+y^2 & xy\end{vmatrix}$$

Now apply R_2-R_1 and R_3-R_1.

$$\Delta=-(x^2+y^2+z^2)\begin{vmatrix}1 & y^2+z^2 & yz\\ 0 & x^2-y^2 & z(x-y)\\ 0 & x^2-z^2 & y(x-z)\end{vmatrix}$$

$$=-(x^2+y^2+z^2)(x-y)(x-z)\begin{vmatrix}1 & y^2+z^2 & yz\\ 0 & x+y & z\\ 0 & x+z & y\end{vmatrix}$$

$$=(x^2+y^2+z^2)(x-y)(z-x)\begin{vmatrix}x+y & z\\ x+z & y\end{vmatrix}$$

Again apply c_1+c_2 and take out the common factor $x+y+z$ from c_1

$$=(x^2+y^2+z^2)(x-y)(z-x)(x+y+z)\begin{vmatrix}1 & z\\ 1 & y\end{vmatrix}$$

$$=(x^2+y^2+z^2)(x+y+z)(x-y)(y-z)(z-x)$$

(k₁) Show that

$$\begin{vmatrix}(b+c)^2 & a^2 & bc\\ (c+a)^2 & b^2 & ca\\ (a+b)^2 & c^2 & ab\end{vmatrix}=(a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a)$$

(Agra 34)

Proceed as above.

***(1) Prove that**

$$\begin{vmatrix}-2a & a+b & c+a\\ b+a & -2b & b+c\\ c+a & c+b & -2c\end{vmatrix}=4(b+c)(c+a)(a+\text{[illegible]}$$

(Alld. 37, Patna 42, Luck. 46 Supp.. Utkal 48)

Expand directly

*(m) $\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = -(x+y+z)(x+y-z)(z+x-y)(y+z-x)$

(Agra 33, Rajputana 52)

Proceeding exactly as in part (h) P. 282 we get

$$\Delta = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = K(x+y+z)(y+z-x)(z+x-y)(x+y-z)\ldots\ldots A$$

It is easy to find K, by comparing the coefficients of x^4 in both sides of A.

Expanding Δ only for x.

$-x \begin{vmatrix} x & z & y \\ y & 0 & x \\ z & x & 0 \end{vmatrix}$ + two other determinants which shall not give x^4

$= -x[\{x(0-x^2)+ - + -\}] = x^4 +$ other terms which shall not be of x^4.

$\therefore$ The coefficient of x^4 is unity where as it is $-K$ on the R. H. S. of A,

$\therefore$ $K = -1$ Hence proved

(n) $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = -(x+y+z)(x+y-z)(z+x-y)(y+z-x)$

(Rajputana 49)

Applying $c_3 - c_4$ and $c_4 - c_3$ we get

$$\Delta = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & -z^2 & x^2-z^2 \\ 1 & y^2 & x^2-y^2 & -y^2 \end{vmatrix} = -\begin{vmatrix} 1 & z^2 & y \\ 1 & -z^2 & x^2-z^2 \\ 1 & x^2-y^2 & -y^2 \end{vmatrix}.$$

Again apply $R_2 - R_1$ and $R_3 - R_1$.

$$\Delta = -\begin{vmatrix} 2z^2 & y^2+z^2-x^2 \\ y^2+z^2-x^2 & 2y^2 \end{vmatrix} = (y^2+z^2-x^2)^2 - 4y^2z^2$$

$$= (y^2+z^2-x^2+2yz)(y^2+z^2-x^2-2yz)$$

$$= \{(y+z)^2 - x^2\}\{(y-z)^2 - x^2\}$$ etc. as given

(o) **Prove that** $\begin{vmatrix} 1 & yz & y+z \\ 1 & zx & z+x \\ 1 & xy & x+y \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}.$

Apply $R_2 - R_1$ and $R_3 - R_1$ on L. H. S. and then on expanding, it comes out to be $(x-y)(y-z)(z-x)$ which is equal to determinant on R. H. S. by ex. 1 Art. 17 Page 252.

Q. 4. Prove that

*(a) $\begin{vmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_3 & -B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$

Where capital letters denote the minor of the corresponding small letters in the determinant on the right.

Denote the two determinants on the left and right by D and D′ respectively then

$$DD'=\begin{vmatrix} a_1A_1-b_1B_1+c_1C_1 & a_2A_1-b_2B_1+c_2C_1 & a_3A_1-b_3A_1+c_3C_1 \\ -a_1A_2+b_1B_2-c_1C_2 & -a_2A_2+b_2B_2-c_2C_2 & -a_3A_2+b_3B_2-c_3C_2 \\ a_1A_3-b_1B_3+c_1C_3 & a_2A_3-b_2B_3+c_2C_3 & a_3A_3-b_3B_3+c_3C_3 \end{vmatrix}$$

which by the result of article no, 11 Page 243.

$$=\begin{vmatrix} D' & 0 & 0 \\ 0 & D' & 0 \\ 0 & 0 & D' \end{vmatrix} = D'^3.$$

$$\therefore \quad DD'=D'^3$$

$$\text{or} \qquad D=D'^2$$

Similarly with the help of article no 12 we can prove

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

(Pb. 31, Madras 38, Annamalai 38, Luck. 42 Supp., Agra 42, Dacca 45, Sagar 49 Aligarh 49)

Where capital letters denote the cofactors of the corresponding small letters of the determinant on the right.

(b$_1$)
$$\begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ca \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix} = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2$$

(Agra 1936, Utkal 46: Benares 48)

Determinant on L. H. S. is of cofactors of the corresponding letters of determinant in the R. H. S

(c) **Show that**

$$\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$$

where $r^2=x^2+y^2+z^2$ and $u^2=yz+zx+xy$

Consider the determinant

$$\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

Cofactor of $x=yz-x^2=X$

Cofactor of $y=-(y^2-xz)=zx-y^2=Y$

Cofactor of $z=xy-z^2=Z$

∴ The given determinant is

$\begin{vmatrix} X & Y & Z \\ Y & Z & X \\ Z & X & Y \end{vmatrix}$ which is a determinant of cofactors of the letters of the determinant $\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$

and therefore by part $b=\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2$

$$=\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$ which on multiplying is

$$=\begin{vmatrix} \Sigma x^2 & \Sigma xy & \Sigma xy \\ \Sigma xy & \Sigma x^2 & \Sigma xy \\ \Sigma xy & \Sigma xy & \Sigma x^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$$

$\because \quad \Sigma x^2 = x^2 + y^2 + z^2 = r^2$

and $\quad \Sigma xy = xy + yz + zx = u^2$

(d) If ω is a cube root of unity then show that square of

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix}$$

(Allahabad 27, Travancore 43, Nagpur 50)

and hence find the value of determinant on the L.H.S.

Replace ω^3 by 1 and remember $1+\omega+\omega^2=0$ Let us denote the determinant in L. H. S. by Δ.

$$\therefore \quad \Delta^2 = \begin{vmatrix} 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega^2 \end{vmatrix} \times \begin{vmatrix} 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega^2 \end{vmatrix}$$

First constituent of first row in product determinant

$$= 1+\omega^2+\omega^4+1 = 1+\omega^2+\omega+1 = 1+0=1.$$

In a similar manner find the other constituent of first row and other rows and prove

$$\Delta^2 = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix}$$

Applying R_2-R_1, R_3+2R_1 and R_4-R_1 and then expanding we find that

$$\Delta^2 = -27$$

$$\therefore \quad \Delta = 3\sqrt{-3}$$

(e) Find the value of

$$\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

where $l_1^2+m_1^2+n_1^2=1$, etc. and $l_1l_2+m_1m_2+n_1n_2=0$, etc.

$$\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$\begin{vmatrix} \Sigma l_1^2 & \Sigma l_1l_2 & \Sigma l_1l_3 \\ \Sigma l_1l_2 & \Sigma l_2^2 & \Sigma l_2l_3 \\ \Sigma l_1l_3 & \Sigma l_2l_3 & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\therefore \quad \Delta = \pm 1$

***(f) Multiply**

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

and show that product of two expressions of the form $L^3+M^3+N^3-3LMN$ is of the same form.

$$\text{Product} = \begin{vmatrix} ax+by+cz & ay+bz+cx & az+bx+cy \\ bx+cy+az & by+cz+ax & bz+cx+ay \\ cx+ay+bz & cy+az+bx & cz+ax+by \end{vmatrix}$$

$$= \begin{vmatrix} X & Y & Z \\ Z & X & Y \\ Y & Z & X \end{vmatrix}$$

$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= -\begin{vmatrix} X & Y & Z \\ Y & Z & X \\ Z & X & Y \end{vmatrix} \quad \text{Where } X = ax+by+cz, \quad Y = ay+bz+cx, \quad Z = az+bx+cy$$

Now $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$ (by expansion)

Hence $(3abc - a^3 - b^3 - c^3)(3xyz - x^3 - y^3 - z^3)$

$= -(3XYZ - X^3 - Y^3 - Z^3)$ Or

$(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$

$= (X^3+Y^3+Z^3-3XYZ)$

i. e. The product of two expressions of the form $L^3+M^3+N^3-3LMN$ is of the same form.

Prove that

(8) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$$

Nagpur 37, Madras 39, Luck. 40, Agra 40, 47, Andhra 41, Allahabad 42, Utkal 45, Travancore 45, 48, Pb. 50, Rajp. 51)

$$\text{L.H.S.} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

If we multiply in the above form then the 1st constituent of the 1st row of product determinant will be $a^2+b^2+c^2$ whereas we want it to be $2bc-a^2$

∴ Re-arranging the terms of 2nd determinant we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times - \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}.$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix},$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

but $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (3abc-a^3-b^3-c^3)^2 = (a^3+b^3+c^3-3abc)^2$

(b) If $ax_1^2+by_1^2+cz_1^2=d=ax_2^2+by_2^2+cz_2^2=ax_3^2+by_3^2+cz_3^2$ and $ax_1x_2+by_1y_2+cz_1z_2=f=ax_2x_3+by_2y_3+cz_2z_3$
$=ax_1x_3+by_1y_3+cz_1z_3$

then prove (Pb. 33)

$$abc\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 = (d-f)^2(d+2f)$$

$$\text{L.H.S.} = abc\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

On multiplying

$$= \begin{vmatrix} \Sigma a x_1^2 & \Sigma a x_1 x_2 & \Sigma a x_1 x_3 \\ \Sigma a x_1 x_2 & \Sigma a x_2^2 & \Sigma a x_2 x_3 \\ \Sigma a x_1 x_3 & \Sigma a x_2 x_3 & \Sigma a x_3^2 \end{vmatrix} \times \begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$$

Now proceed as in part (n) of Q. 2 page 270.

([illegible]) Express

$$\triangle = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

(Agra 32, 37, 49, 51, Andhra 38, I. A. A. S. 40)

as product of two determinants and find its value.

If we put $x=y$ in $\triangle$ then $c_1=c_2$ $\therefore$ $\triangle=0$.

Hence $x-y$ is a factor.

Similarly $y-z, z-x$ are also factors of $\triangle$.

but $(x-y)(y-z)(z-x) = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ Ex. 1 Art. 17 page 252

Again if we put $a=b$ in $\triangle$ then $R_1=R_2$ $\therefore$ $\triangle=0$

Hence $(a-b)$ is a factor.

Similarly $(b-c)$, $(c-a)$ are also factors of $\triangle$. But

$(a-b)(b-c)(c-a) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ Ex. 1 Art. 17 page 252

$$\therefore \triangle = K \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

For some such suitable value of K so that on multiplying the two determinants the resulting determinant is Δ.

$$\therefore \quad \Delta = K\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad \ldots\ldots(A)$$

$$= -K\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$\because$ we have interchanged two columns

If we multiply we get 1st constituent of product determinant as x^2+ax+a^2 whereas we want it to be $x^2-2ax+a^2$.

$\therefore$ K must be chosen$=2$

$$\therefore \quad \Delta = \begin{vmatrix} x^2 & -2x & 1 \\ y^2 & -2y & 1 \\ z^2 & -2z & 1 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

is required form, because on multiplication it becomes equal to Δ.

$$\text{The value of } \Delta = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$=2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$$

(j) **Express**

$$\Delta = \begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix} \qquad \text{(Agra '35)}$$

as product of two determinants and find its value.

Proceed like (i) part and

$\Delta=$(A) form of (i) part when $K=2$

and its value is $2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

(k) Find the condition in determinant form, in order that the general expression of the second degree in two variables $f(x, y)=ax^2+2hxy+by^2+2gx+2fy+c$ may be resolved into two linear factors.

Let the two linear factors be

$$l_1x+m_1y+n_1 \text{ and } l_2x+m_2y+n_2$$

then $f(x,y)=(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)$

Comparing the coefficients.

$$l_1l_2=a,\ m_1m_2=b,\ n_1n_2=c$$

$$l_1m_2+l_2m_1=2h,\ m_1n_2+m_2n_1=2f,\ n_1l_2+n_2l_1=2g$$

Now we know that $\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix}=0$ and $\begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix}=0$

Multiply these two zero determinants we get

$$\begin{vmatrix} 2l_1l_2 & l_1m_2+l_2m_1 & n_1l_2+n_2l_1 \\ m_1l_2+m_2l_1 & 2m_1m_2 & m_1n_2+m_2n_1 \\ n_1l_2+n_2l_1 & m_1n_2+m_2n_1 & 2n_1n_2 \end{vmatrix}=0$$

or $\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix}=0$

or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$ is the required condition

On expanding it becomes

$$abc+2fgh-af^2-bg^2-ch^2=0$$

i. e. the condition that general equation of second degree may represent two straight lines,

(l) **Prove that**

$$\Delta=\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}=0$$

we know that $\begin{vmatrix} 0 & 1 & 1 \\ 0 & \gamma+\delta & \alpha+\beta \\ 0 & \gamma\delta & \alpha\beta \end{vmatrix}=0$ and $\begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha+\beta & \gamma+\delta \\ 0 & \alpha\beta & \gamma\delta \end{vmatrix}=0$

on multiplying these two zero determinants we get Δ which therefore is also equal to zero.

(m) **Prove that** **(Agra 28, 44, Travancore 47)**

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}\times\begin{vmatrix} \alpha-i\beta & \gamma-i\delta \\ -\gamma-i\delta & \alpha+i\beta \end{vmatrix}$$

can be written in the form $\begin{vmatrix} A-iB & C-iD \\ -C-iD & A+iB \end{vmatrix}$

On multiplying, the first constituent of first row in product determinant will be

$$(a+ib)(\alpha-i\beta)+(c+id)(\gamma-i\delta)$$
$$=\underbrace{a\alpha+b\beta+c\gamma+d\delta}_{}\quad -i\underbrace{(a\beta-b\alpha+c\delta-d\gamma)}_{}$$
$$=\quad A \qquad -i \quad B$$

Similarly 2nd constituent

$$(a+ib)(-\gamma-i\delta)+(c+id)(\alpha+i\beta)$$
$$=\underbrace{-a\gamma+b\delta+c\alpha-d\beta}_{}\quad -i\underbrace{(a\delta+b\gamma-c\beta-d\alpha)}_{}$$
$$=\quad C \qquad -i \quad D$$

Similarly the constituents of 2nd row can be easily found out to be

C $-i$D and A$+i$B

$$\therefore \begin{vmatrix} a+ib & c+id \\ c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha-i\beta & \gamma-i\delta \\ -\gamma-i\delta & \alpha+i\beta \end{vmatrix} = \begin{vmatrix} A-iB & C-iD \\ -C-iD & A+iB \end{vmatrix}$$

where A, B, C and D have the values written above.

Expanding these determinants we get

$$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=A^2+B^2+C^2+D^2$$

which shows that the product of two sums each of four squares can be expressed as the sum of four squares.

Hence express

$$(9^2+2^2+3^2+4^2)(5^2+6^2+7^2+8^2)$$

as the sum of four squares.

Let $a=9$, $b=2$, $c=3$, $d=4$ and $\alpha=5$, $\beta=6$, $\gamma=7$, $\delta=8$.

Then $A=a\alpha+b\beta+c\gamma+d\delta=110$.

Similarly $B=40$, $C=56$ and $D=48$.

$$\therefore \quad (9^2+2^2+3^2+4^2)(5^2+6^2+7^2+8^2)$$
$$=(110^2+40^2+56^2+48^2)$$

(n) **Express as a determinant**

$$\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

$$=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$=\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

(o) **Prove that**

$$\begin{vmatrix} x^2+y^2+a^2 & 2ax+xy & 2ay+x^2 \\ 2ax+xy & a^2+2x^2 & 2ax+xy \\ 2ay+x^2 & 2ax+xy & x^2+y^2+a^2 \end{vmatrix} = \begin{vmatrix} a & x & y \\ x & a & x \\ y & x & a \end{vmatrix}^2 ;$$

After multiplying $\begin{vmatrix} a & x & y \\ x & a & x \\ y & x & a \end{vmatrix}$ by itself we get determinant on L. H. S.

Hence proved.

(p) **Prove that** $\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$

is a multiple of the determinant.

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **and find the other factor.**

(Pb. 46)

It is easy to see that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} \text{ etc.}$$

hence the other factor is $\begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix}$ i. e. 13

(q) **Prove that**

$$\begin{vmatrix} b^2+c^2+1 & c^2+1 & b^2+1 & b+c \\ c^2+1 & c^2+a^2+1 & a^2+1 & c+a \\ b^2+1 & a^2+1 & a^2+b^2+1 & a+b \\ b+c & c+a & a+b & 3 \end{vmatrix} = (bc+ca+ab)^2$$

(Rajputana 50)

By inspection we find that

$$\Delta = \begin{vmatrix} 0 & b & c & 1 \\ a & 0 & c & 1 \\ a & b & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 0 & b & c & 1 \\ a & 0 & c & 1 \\ a & b & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \Delta'^2$$

where Δ' is easily seen by applying $C_2 - bC_4$, $C_3 - cC_4$ and expanding, to be equal to $(bc+ca+ab)^2$

Q. 5. Solve the equations.

(a)

$$\Delta = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0$$

Put $x=a$ and b and you find that Δ vanishes because of identical lines. Hence a and b are the roots of Δ which is of second degree in x.

(b)

$$\Delta = \begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$$

Apply $R_1 - 2R_3$, and $R_2 - 3R_3$

$$\Delta = \begin{vmatrix} 1 & -17 & -16 \\ -10 & -25 & -23 \\ 7-x & 14 & 13 \end{vmatrix} = 0$$

Again apply C_3-C_2 and then make two zeros in C_3 of the determinant and expanding we get $x=4$.

(c) $$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}=0$$ (Agra 35, 51, Luck. 48)

Apply R_2-R_1, and R_3-R_1, we get.

$$\Delta=\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix}=0$$

$$\therefore \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}=0$$

Apply C_2-2C_1 and C_3-3C_1 and then make two zeros in column No 2 of the new determinant and expand we get $x=4$.

*(d) $$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix}=0$$ (Agra 33, Luck. 50)

Apply $C_2-\frac{3}{2}C_1$ and C_3-2C_1

$$\Delta=\begin{vmatrix} 4x & 2 & 1 \\ 6x+2 & 0 & -4 \\ 8x+1 & -\frac{3}{2} & 0 \end{vmatrix}=0$$

Now expanding with third column we find $x=-\frac{11}{97}$

*(e) $$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix}=0$$ (Pb. 24

Apply R_2-2R_1, R_3-3R_1 and then make one zero in column I of the new determinant thus obtained and expand. we get $x=-2,-1,-1$.

*(f) Show that $x=2$ is a root of

$$\Delta=\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}=0$$ and solve it completely

(Agra 48, Raj. 48)

If we put $x=2$ then $R_1=R_2$. $\therefore$ $\Delta=0$.

Hence $x=2$ is a root of $\Delta=0$.

Expanding we get $\Delta=-5x^3+35x-30=0$ or $x^3\ 7x+6=0$ or $(x-1)(x+3)(x-2)=0$ $\therefore$ other roots are 1 and -3.

(g) $x+y+z=1$ $\therefore$ By art. No. 19,

$ax+by+cz=K$

$a^2x+b^2y+c^2z=K^2$

$$x.\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ K & b & c \\ K^2 & b^2 & c^2 \end{vmatrix}$$

$\therefore$ $x(a-b)(b-c)(c-a)=(K-b)(b-c)(c-K)$ [Ex 1 § 17]

$$\therefore\ x=\frac{(K-b)(b-c)(c-K)}{(a-b)(b-c)(c-a)}=\frac{(K-b)(c-K)}{(a-b)(c-a)} \qquad (P\ 252)$$

$$=\frac{(K-b)(K-c)}{(a-b)(a-c)}$$ Similarly write the values of y and z by symmetry.

(h) $ax+by+cz=K$

$a^2x+b^2y+c^2z=K^2$ By article no. 19.

$a^3x+b^3y+c^3z=K^3$

$$x.\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=\begin{vmatrix} K & b & c \\ K^2 & b^2 & c^2 \\ K^3 & b^3 & c^3 \end{vmatrix}$$

or $xabc(a-b)(b-c)(c-a)=Kbc(K-b)(b-c)(c-K)$ (Ex. 1 Art. 17)

$\therefore\ x=\dfrac{(K-b)(K-c)}{a(a-b)(a-c)}$. Similarly write the values of y and z by symmetry.

(i)
$$\left.\begin{aligned} x+y+z+u&=1\\ ax+by+cz+du&=K\\ a^2x+b^2y+c^2z+d^2u&=K^2\\ a^3x+b^3y+c^3z+d^3u&=K^3\end{aligned}\right\}\ \text{By article 19.}$$

$$u.\begin{vmatrix} 1 & 1 & 1 & 1\\ a & b & c & d\\ a^2 & b^2 & c^2 & d^2\\ a^3 & b^3 & c^3 & d^3\end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 & 1\\ a & b & c & K\\ a^2 & b^2 & c^2 & K^2\\ a^3 & b^3 & c^3 & K^3\end{vmatrix}$$ ∴ By Q. 2 part (d) Page 280

or $u(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$

$=(a-b)(a-c)(a-K)(b-c)(b-K)(c-K)$

$\therefore\ u=\dfrac{(a-K)(b-K)(c-K)}{(a-d)(b-d)(c-d)}$ or $\dfrac{(K-a)(K-b)(K-c)}{(d-a)(d-b)(d-c)}$

Similarly write down the values of x, y and z by symmetry

(j)
$$\begin{aligned} x+y+z+u&=K\\ ax+by+cz+du&=K^2\\ a^2x+b^2y+c^2z+d^2u&=K^3\\ a^3x+b^3y+c^3z+d^3u&=K^4\end{aligned}$$

Proceeding as above $u=\dfrac{K(K-a)(K-b)(K-c)}{(d-a)(d-b)(d-c)}$

(k)
$$\begin{aligned} x+2y+3z&=6\\ 2x+4y+z&=7\\ 3x+2y+9z&=14\end{aligned}\qquad [x=y=z=1]$$

(l) If $(f^2-bc)x+(ch-fg)y+(bg-hf)z=0$……(1)

$(ch-fg)x+(g^2-ca)y+(af-gh)z=0$……(2)

$$(bg-hf)x+(af-gh)y+(h^2-ab)z=0\ldots\ldots(3)$$

then show that $abc+2fgh-af^2-bg^2-ch^2=0$.

In order that the three given equations may hold, then by article no. 2 the eliminant

$$\begin{vmatrix} f^2-bc & ch-fg & bg-hf \\ ch-fg & g^2-ca & af-gh \\ bg-hf & af-gh & h^2-ab \end{vmatrix}=0\ldots\quad(A)$$

Consider $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Minor of $a=A=bc-f^2$. Similarly $B=ac-g^2$, $C=ab-h^2$, $F=af-gh$, $G=hf-bg$, and $H=ch-fg$.

Hence determinant A becomes

$$\begin{vmatrix} -A & H & -G \\ H & -B & F \\ -G & F & -C \end{vmatrix}=0$$

or $\begin{vmatrix} A & -H & G \\ -H & B & -F \\ G & -F & C \end{vmatrix}=0$ or by Q. 4 (a) P. 288

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2=0 \quad\therefore\quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Expanding $abc+2fgh-af^2-bg^2-ch^2=0$.

(n) Find the condition that $lx+my+nz=0$ may be fied by (a_r, b_r, c_r) where $r=1, 2, 3$ and show that it

is the same as the condition that the three equations $a_rx+b_ry+c_rz=0$ may be satisfied by (l, m, n).

Since (a_r, b_r, c_r) for $r=1, 2, 3$ satisfies $lx+my+nz=0$.

$$\therefore\quad \begin{aligned} a_1l+b_1m+c_1n&=0\\ a_2l+b_2m+c_2n&=0\\ a_3l+b_3m+c_3n&=0 \end{aligned}$$

Eliminating l, m, n by article no. 2 we get the required condition as

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{......(A)}$$

Again since $a_rx+b_ry+c_rz=0$ where $r=1, 2, 3$ are satisfied by l, m, n.

$$\therefore\quad \begin{aligned} a_1l+b_1m+c_1n&=0\\ a_2l+b_2m+c_2n&=0\\ a_3l+b_3m+c_3n&=0 \end{aligned}$$

Again eliminating l, m, n by article no. 2, we get the condition as

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

which is the same as condition (A)

(n) Find λ when

$$\begin{aligned} ax+hy+g&=0\\ hx+by+f&=0\\ gx+fy+c&=\lambda \end{aligned}$$

or

$$\begin{aligned} ax+hy+g&=0\\ hx+by+f&=0\\ gx+fy+c-\lambda&=0 \end{aligned}$$

Eliminating x, y from above set of equations we get

$$\begin{vmatrix} a & h & g+0 \\ h & b & f+0 \\ g & f & c-\lambda \end{vmatrix}=0$$

or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} + \begin{vmatrix} a & h & 0 \\ h & q & 0 \\ g & f & -\lambda \end{vmatrix}=0$$

$$(abc+2fgh-af^2-bg^2-ch^2)-\lambda(ab-h^2)=0$$

$$\therefore \quad \lambda=\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}$$

(p) **Find K where**

$$l_1x+m_1y+n_1z+p_1=k$$
$$l_2x+m_2y+n_2z+p_2=0$$
$$l_3x+m_3y+n_3z+p_3=0$$
$$l_4x+m_4y+n_4z+p_4=0$$

Procced as above

$$K=-\begin{vmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{vmatrix} \div \begin{vmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_4 & m_4 & n_4 \end{vmatrix}$$

***(q) Eliminate a, b and c from**

$$x=\frac{a}{b-c},\ y=\frac{b}{c-a},\ z=\frac{c}{a-b} \qquad \text{(Pb. 36)}$$

$$\therefore \begin{cases} a-bx+cx=0 \\ -ay-b+cy=0 \\ az-bz-c=0 \end{cases}$$

Eliminating a, b, c by article 2, we get

$$\begin{vmatrix} 1 & -x & x \\ -y & -1 & y \\ z & -z & -1 \end{vmatrix}=0$$

Expanding we get

$$1+xy+yz+zx=0$$

*(r) **Having given the equations**

$$x=cy+bz,\ y=az+cx,\ z=bx+ay$$

where x, y, z are not all zero, prove that

$$a^2+b^2+c^2+2abc=1$$

$$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2} \qquad \text{(Pb. 45)}$$

Rewriting the above equations & eliminating x, y, z we get

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}=0$$

Or $a^2+b^2+c^2+2abc=1$ (on expansion) I

Again solving the first two equations by the method of Cross multiplication we get

$$\frac{x}{-ac-b}=\frac{y}{-bc-a}=\frac{z}{-1+c^2} \text{ Squaring}$$

$$\frac{x^2}{a^2c^2+b^2+2abc}=\frac{y^2}{a^2+b^2c^2+2abc}=\frac{z^2}{(1-c^2)^2}$$

Using relation (1) we get

$$\text{Or } \frac{x^2}{1-a^2-c^2+a^2c^2}=\frac{y^2}{1-b^2-c^2+b^2c^2}=\frac{z^2}{(1-c^2)^2}$$

$$\text{Or } \frac{x^2}{(1-a^2)(1-c^2)}=\frac{y^2}{(1-b^2)(1-c^2)}=\frac{z^2}{(1-c^2)^2}$$

Cancel $(1-c^2)$ throughout from the denominator we get the reqd. result.

CHAPTER VI

BINOMIAL THEOREM

§1. **Definition.** An algebaric expression of two terms is called a binomial. Thus $3x+4y$, $7x^3-5y^3$, $x+a$, are all binomials.

The formula

$$(x+a)^n=x^n+{}^nc_1x^{n-1}a+{}^nc_2x^{n-2}a^2+{}^nc_rx^{n-r}a^r+\ldots+a^n$$

where n is a +ive integer is called the **Binomial Theorem** for +ive integral index.

§2. Proof of Binomial Theorem.

By actual multiplication we find that

$$(x+a)^2=x^2+2ax+a^2=x^2+{}^2c_1ax+{}^2c_2a^2$$

$$(x+a)^3=x^3+3ax^2+3a^2x+a^3=x^3+{}^3c_1ax^2+{}^3c_2a^2x+{}^3c_3a^3$$

$$(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4=x^4+{}^4c_1ax^3$$
$$+{}^4c_2a^2x^2+{}^4c_3a^3x+{}^4c_4a^4$$

In the light of above let us assume that

$$(x+a)^n=x^n+{}^nc_1x^{n-1}a+{}^nc_2x^{n-2}a^2+\ldots\ldots+{}^nc_rx^{n-r}a^r$$
$$+\ldots\ldots+{}^nc_na^n$$

Multiplying both sides by $(x+a)$ we get

$$(x+a)^{n+1}=(x+a)(x^n+{}^nc_1x^{n-1}a+{}^nc_2x^{n-2}a^2+\ldots\ldots$$
$$+{}^nc_rx^{n-r}a^r+\ldots\ldots+{}^nc_na^n$$

$$=x^{n+1}+({}^nc_1+1)x^n.a+({}^nc_2+{}^nc_1)x^{n-1}.a^2+\ldots\ldots$$
$$+({}^nc_r+{}^nc_{r-1})x^{n-r+1}a^r+\ldots\ldots{}^nc_na^{n+1}$$

we know that ${}^nc_r+{}^nc_{r-1}={}^{n+1}c_r$

Putting $r=1, 2, 3\ldots\ldots$

$${}^nc_1+{}^nc_0={}^nc_1+1={}^{n+1}c_1$$

$${}^nc_2+{}^nc_1={}^{n+1}c_2$$

$${}^nc_3+{}^nc_2={}^{n+1}c_3$$

$${}^nc_r+{}^nc_{r-1}={}^{n+1}c_r$$

and $^nc_n = {}^{n+1}c_{n+1} = 1$

∴ $(x+a)^{n+1} = x^{n+1} + {}^{n+1}c_1 x^n . a + {}^{n+1}c_2 x^{n-1} . a^2$
$+ {}^{n+1}c_r x^{n-r+1} \; a^r + \ldots\ldots + {}^{n+1}c_{n+1} a^{n+1}$

we find that the form of the above expansion is the same as that of $(x+a)^n$ which we assumed. Hence our assumption is true for $n = n+1$.

But we find that when $n = 2, 3, 4$ and so on the above expansion is true. Hence it is true for all positive integral values of n.

§3. Some important features of Binomial expansion.

$$(x \quad a)^n = x^n + {}^nc_1 x^{n-1} a + {}^nc_2 x^{n-2} a^2 + \ldots + {}^nc_r x^{n-r} a^r + \ldots + a^n$$

(a) $T_2 = {}^nc_1 x^{n-1} . a$ $\quad T_3 = {}^nc_2 x^{n-2} a^2$

$T_4 = {}^nc_3 x^{n-3} a^3$ and so on.

∴ $T_{r+1} = {}^nc_r x^{n-r} a^r$ which we shall call the **general term** of the binomial expansion.

(b) We know that ${}^nc_r = {}^nc_{n-r}$. Put $r = 1, 2, 3.$

i. e. ${}^nc_1 = {}^nc_{n-1}$

${}^nc_2 = {}^nc_{n-2}$ and so on

${}^nc_r = {}^nc_{n-r}$.

But we find that each group $({}^nc_1, {}^nc_{n-1}), ({}^nc_2, {}^nc_{n-2})$ $\ldots\ldots({}^nc_r, {}^nc_{n-r})$ contains the coefficients of terms equidistant from the beginning and the end and hence they are equal.

∴ **The coefficients of terms equidistant from the beginning and the end are equal.**

$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$, $\;(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$

(c) **The number of terms is one more than the index of the binomial.**

(d) Putting $a = -a$ in the expansion of $(x+a)^n$, we get

$$(x-a)^n = x^n - {}^nc_1 x^{n-1} a + {}^nc_2 x^{n-2} a^2 - \ldots\ldots + \ldots (-1)^r x^{n-r} a^r + \ldots\ldots + (-1)^n a^n$$

We observe that the terms are alternately +ive and −ive but numerically are the same as those in $(x+a)^n$

§ 4. Binomial coefficients.

$$(1+x)^n = 1 + {}^nc_1 x + {}^nc_2 x^2 + \ldots\ldots {}^nc_r x^r + \ldots\ldots {}^nc_n x^n$$
$$= c_0 + c_1 x + c_2 x^2 + \ldots\ldots c_r x^r + \ldots\ldots c_n x^n \ldots\ldots (A)$$

$\because$ $c_0 - {}^nc_0 = 1$ where c_r means ${}^nc_r$.

$c_0, c_1, c_2, \ldots\ldots c_n$ are called the binomial coefficients.

Properties of binomial coefficients.

(a) The sum of the binomial coefficients $= 2^n$.

In the above expansion (A) put $x=1$ and we get

$2^n = c_0 + c_1 + c_2 + c_3 + \ldots\ldots c_n$ which proves the theorem

Also by taking c_0 i. e. 1 on the left hand side, we get

$$2^n - 1 = c_1 + c_2 + c_3 + \ldots\ldots c_n.$$

(b) The sum of the odd coefficients is equal to the sum of even coefficients and each sum is equal to 2^{n-1}:

In the above expansion (A) put $x = -1$ and we get

$$0 = c_0 - c_1 + c_2 - c_3 + c_4 - c_5 \ldots\ldots\ldots$$

$$\therefore \underbrace{c_1 + c_3 + c_5 + \ldots\ldots}_{(x)} = \underbrace{c_0 + c_2 + c_4 + c_6 + \ldots\ldots}_{(y)}$$

$\therefore$ $x + y =$ sum of all the binomial coefficients $= 2^n$ by (a)

or $2x = 2^n$ $\quad\therefore$ $x = y = \frac{2^n}{2} = 2^{n-1}$.

(c) Greatest binomial coefficient

Students must have read in the chapter of permutation and combination in intermediate that if n be even then ${}^nc_r$ is greatest when $r = \frac{n}{2}$ and if n be odd then ${}^nc_r$ is greatest when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

Hence when n is given greatest binomial coefficient is ${}^nc_{n/2}$ i.e. $c_{n/2}$ and when n is odd, then it is ${}^nc_{(n-1)/2}$ or ${}^nc_{(n+1)/2}$ both being equal.

Ex. 1. If c_0, c_1, $c_2 \ldots\ldots c_n$ denote the binomial coefficients in the expansion of $(1+x)^n$, prove that

(i) $c_1+2c_2+3c_3+\ldots n.cn=n\,.\,2^{n-1}$

(ii) $c_1-2c_2+3c_3+\ldots\ldots+(-1)^n\,n\,.\,c_n=0$

(iii) $c_0+2c_1+3c_2+\ldots\ldots+(n+1)c_n=(n+2)2^{n-1}$

(Mysore 32, Rangoon 50)

(iv) $c_0-2c_1+3c_2\ldots\ldots+(-1)^n(n+1)c_n=0$

(Patna 21)

(i) $\text{L.H.S.}=n+2\,.\,\dfrac{n(n-1)}{2\,!}+3\,.\,\dfrac{n(n-1)(n-2)}{3\,!}+\ldots\ldots$

$$=n\left\{1+(n-1)+\frac{(n-1)(n-2)}{2\,!}+\ldots\ldots\right\}$$

Put $n-1=\text{N}$ in the term within brackets.

$$=n\left\{1+\text{N}+\frac{\text{N}(\text{N}-1)}{2\,!}+\ldots\ldots\right\}$$

$$=n\{{}_{\text{N}}c_0+{}_{\text{N}}c_1+{}_{\text{N}}c_2+\ldots\ldots\}$$

$=n\,.\,2^{\text{N}}$ by $(4\,.\,a)=n\,.\,2^{n-1}$

(ii) Proceeding as above $\text{L.H.S.}=n[{}_{\text{N}}c_0-{}_{\text{N}}c_1+{}_{\text{N}}c_2-{}_{\text{N}}c_3\ldots]$

$=n\,.\,0=0$ by (§4.b)

(iii) $\text{L.H.S.}=[c_0+c_1+c_2+\ldots\ldots c_n]+[c_1+2c_2+3c_3+\ldots n\,.\,c_n]$

$=2^n+n\,.\,2^{n-1}$ (from i)$=2^{n-1}[2+n]$

(iv) Proceed as above

Ex. 2. $\dfrac{c_1}{c_0}+\dfrac{2c_2}{c_1}+\dfrac{3c_3}{c_2}+\ldots\ldots\dfrac{n.c_n}{c_{n-1}}=\dfrac{n(n+1)}{2}$

$$\text{L.H.S.}=\frac{n}{1}+2\,.\,\frac{\dfrac{n(n-1)}{2\,!}}{n}+3\,.\,\frac{\dfrac{n(n-1)(n-2)}{3\,!}}{\dfrac{n(n-1)}{2}}+\ldots\ldots$$

$=n+(n-1)+(n-2)+\ldots\ldots+3+2+1$

$=$sum of n natural numbers$=\dfrac{n(n+1)}{2}$

Ex. 3. (i) $c_0+\frac{c_1}{2}+\frac{c_2}{3}+\dots\dots\frac{c_n}{n+1}=\frac{2^{n+1}-1}{n+1}$ **(Delhi 50)**

(ii) $c_0-\frac{c_1}{2}+\frac{c_2}{3}-\dots\dots+(-1)^n\ \frac{c_n}{n+1}=\frac{1}{n+1}$

(i) L.H.S. $=\left\{1+\frac{n}{2}+\frac{n(n-1)}{3\,.\,(2)\,!}+\frac{n(n-1)(n-2)}{4\,.\,(3)\,!}+\dots\dots\right\}$

$$=\frac{1}{n+1}\left\{(n+1)+\frac{(n+1)\,.\,n}{2\,!}+\frac{(n+1)\,.\,n(n-1)}{3\,!}+\frac{(n+1)\,.\,n(n-1)(n-2)}{4\,!}+\dots\dots\right\}$$

Put $n+1=\text{N}$ in the terms within brackets

$$=\frac{1}{n+1}\left\{\text{N}+\frac{\text{N}(\text{N}-1)}{2\,!}+\frac{\text{N}(\text{N}-1)(\text{N}-2)}{3\,!}+\frac{\text{N}(\text{N}-1)(\text{N}-2)(\text{N}-3)}{4\,!}+\dots\right\}$$

$$=\frac{1}{n+1}\left\{\text{N}c_1+\text{N}c_2+\text{N}c_3+\text{N}c_4+\dots\dots\right\}$$

$=\frac{1}{n+1}\left\{2^{\text{N}}-1\right\}$ ∴ $\text{N}c_0$ i. e. 1 is not present in the bracket

$=\frac{1}{n+1}\left\{2^{n+1}-1\right\}$ ∵ $\text{N}=n+1$

(ii) Proceeding as above L.H.S. $=\frac{1}{n+1}\left[\text{N}c_1-\text{N}c_2+\text{N}c_3-\text{N}c\ \dots\dots\dots\right]=\frac{1}{n+1}$ (§4.b)

Ex. 4. $\frac{c_0}{1}+\frac{c_2}{3}+\frac{c_4}{5}+\dots=\frac{2^n}{n+1}$

L.H.S. $=1+\frac{n(n-1)}{3\,(2)\,!}+\frac{n(n-1)(n-2)(n-3)}{5\,(4)\,!}+\dots\dots$

$$=\frac{1}{n+1}\left\{(n+1)+\frac{(n+1)n(n-1)}{3\,!}+\frac{(n+1).n(n-1)(n-2)(n-3)}{5\,!}+\ldots\ldots\right\}$$

Put $n+1=\text{N}$ in the terms within brackets.

$$=\frac{1}{n+1}\{\text{N}c_1+\text{N}c_3+\text{N}c_5+\ldots\ldots\}$$

$$\frac{1}{n+1}\left\{2^{\text{N}-1}\right\}\text{ by §4. } b=\frac{1}{n+1}\{2^n\} \because \text{N}+1=n$$

Ex. 5. (i) $c_0{}^2+c_1{}^2+c_2{}^2+\ldots c_n{}^2=\dfrac{(2n)\,!}{n\,!.n\,!}$

(Allahabad 1904)

(ii) $c_0{}^2-c_1{}^2+c_2{}^2-c_3{}^2+\ldots+(-1)^n c_n{}^2=0$

or $(-1)^{n/2}\dfrac{n\,!}{(n/2)\,!\,(n/2)\,!}$ according as n is odd or even

(iii) $c_0c_n+c_1c_{n-1}+\ldots.c_nc_0=\dfrac{(2n)\,!}{n\,!.n\,!}$

(iv) $c_0c_1+c_1c_2+c_2c_3+c_{n-1}c_n=\dfrac{(2n)\,!}{(n-1)\,!.(n+1)\,!}$

(v) $c_0c_2+c_1c_3+c_2c_4+\ldots\ldots c_{n-2}c_n=\dfrac{(2n)\,!}{(n+2)!.(n-2)!}$

(vi) $c_0c_r+c_1c_{r+1}\ldots\ldots c_{n-r}.c_n=\dfrac{(2n)\,!}{(n-r)\,!.(n+r)\,!}$

(Madras 1904)

$$(1+x)^n=c_0+c_1x+c_2x^2+\ldots\ldots+c_{n-1}x^{n-1}+c_nx^n\ldots1$$

Putting $x=1/x$ in both sides of above we get

$$\left(1+\frac{1}{x}\right)^n=c_0+c_1.\frac{1}{x}+c_2\frac{1}{x^2}+\ldots\ldots c_{n-1}.\frac{1}{x^{n-1}}$$
$$+c_n.\frac{1}{x^n}\ldots2$$

Multiplying 1 and 2 we get

$$[c_0+c_1x+c_2x^2+\ldots\ldots c_{n-1}x^{n-1}+c_nx^n]$$
$$\left[c_0+c_1.\frac{1}{x}+c_2.\frac{1}{x^2}+\ldots\ldots c_{n-1}.\frac{1}{x^{n-1}}+c_n.\frac{1}{x^n}\right]$$
$$=(1+x)^n.\left(\frac{1+x}{x}\right)^n=\frac{1}{x^n}(1+x)^{2n}\ldots\ldots3$$

I. Clearly $c_0^2+c_1^2+c_2^2+\ldots+c_n^2=$ the term independent of x in L. H. S. of (3) = term independent of x in R.H.S. i.e. in $\frac{1}{x^n}(1+x)^{2n}$ or the coefficient of x^n in $(1+x)^{2n}$ which is clearly ${}^{2n}c_n=\frac{(2n)!}{n!.n!}$

(iii) We know $c_n=c_0$, $c_{n-1}=c_1$ etc. and hence it is same as part 1.

VI. Clearly $c_0c_r+c_1c_{r+1}+c_2c_{r+2}\ldots=$ coefficient of x^r in L.H.S. of (3) = coefficient of x^r in R.H.S. i.e. in $\frac{1}{x^n}(1+x)^{2n}$ or the coefficient of x^{n-r} in $(1+x)^{2n}$ which is ${}^{2n}c_{n-r}$

$$=\frac{(2n)!}{(n-r)!.(n+r)!}$$

IV and V Putting $r=1$ and 2 in VI we get the required results.

(ii) $c_0^2-c_1^2+c_2^2-c_3^2+\ldots+(-1)^nc_n^2=$ term independent of x in the product of $(1-x)^n\left(1+\frac{1}{x}\right)^n$ or in $(1-x)^n\frac{(1+x)^n}{x^n}$ or in $\frac{1}{x^n}(1-x^2)^n$ or = coefficient of x^n in $(1-x^2)^n$

Now we have to find the term of x^n in $(1-x^2)^n$

Case 1. n is odd. Since the expansion of $(1-x^2)^n$ will involve only even powers and hence there will be no term of x^n and as such its coefficient is zero.

Case II. n even $x^n=(x^2)^{n/2}$ and hence the required term in the expansion of $(1-x^2)^n$ is ${}^nc_{n/2}(-x^2)^{n/2}$

$=\frac{n!}{(n/2)!.(n/2)!}(-1)^{n/2}.x^n$ and the coefficient is

$(-1)^{n/2}\dfrac{n!}{(n/2)!\,(n/2)!}$ is the required sum.

Ex. 6. If n be an even integer, prove that

$$\frac{1}{1!\,(n-1)!}+\frac{1}{3!\,(n-3)!}+\frac{1}{5!\,(n-5)!}+\ldots\ldots\frac{1}{(n-1)!\,1!}=\frac{2^{n-1}}{n!}$$

$$\text{L. H. S,}=\frac{1}{n!}\left[\frac{n!}{1!\,(n-1)!}+\frac{n!}{3!\,(n-3)!}+\frac{n!}{5!\,(n-5)!}+\ldots\ldots\frac{n!}{(n-1)!\,1!}\right]$$

$$=\frac{1}{n!}\left[\frac{n}{1!}+\frac{n(n-1)(n-2)}{3!}+\ldots\right]=\frac{1}{n!}[c_1+c_3+c_5+\ldots]$$

$$=\frac{1}{n!}.2^{n-1}\qquad(\S\ 4a)$$

§ 5. Proof of Binomial Theorem for any index.

(Agra 31, 33)

We have proved that when n is a +ive integer, then

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots\ldots(1)$$

Now let n be +ive, −ive, fractional or integral and $f(n)$ stand for $1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots\ldots$

$$\therefore\quad f(n)\,f(m)=\left(1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots\ldots\right)\left(1+mx+\frac{m(m-1)}{2!}x^2+\ldots\ldots\ldots\right)$$

$$=1+(m+n)x+\frac{x^2}{2!}[n(n-1)+m(m-1)+2mn]+\ldots\ldots$$

$$=1+(m+n)x+\frac{x^2}{2!}[m^2+n^2+2mn-(m+n)]+\ldots\ldots$$

$$=1+(m+n)x+\frac{x^2}{2!}[(m+n)^2-(m+n)]+\ldots\ldots$$

$$=1+(m+n)x+(m+n)(m+n-1)\frac{x^2}{2!}+\ldots\ldots$$

Thus the product of the two series gives rise to another series in ascending powers of x which is exactly of the same form as $f(m)$ or $f(n)$ and whose coefficients will not change in form for all values of m and n. Therefore to determine the form of these coefficients we may give any values to m and n that are most convenient to us and let such values be +ive integers, then in that case.

$$f(n)=1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots=(1+x)^n \text{ from (1)}$$

$$f(m)=1+mx+\frac{m(m-1)}{2!}x^2+\ldots\ldots=(1+x)^m \text{ from (1)}$$

$$\therefore\ f(m).f(n)=(1+x)^m.(1+x)^n$$

$$=1+(m+n)x+\frac{(m+n)(m+n-1)}{2!}x^2+\ldots\ldots$$

$=f(m+n)$ because when m and n are +ive integers, hence $m+n$ is also a +ive integer.

Hence for all values of m and $n, f(m) \,.\, f(n)=f(m+n)$

Similarly $f(m) \,.\, f(n) \,.\, f(p)=f(m+n) \,.\, f(p)=f(m+n+p)$

Case 1. +ive fractional index.

Let $n=p/q$ where p and q are +ive integers, then

$$f(p/q) \,.\, f(p/q) \,.\, f(p/q)\ldots\text{to } q \text{ factors}=f(p/q+p/q+\ldots q \text{ times})$$
$$=f(q \,.\, p/q)=f(p)$$

$$1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+\ldots\ldots\ldots$$

$=(1+x)^p \qquad \therefore\ [f(p/q)]^q=(1+x)^p$

Taking qth root of both sides, we get

$$f(p/q)=(1+x)^{p/q}$$

Case II. Negative Index.

We have proved that $f(m)\times f(n)=f(m+n)$ for all values of m & n. Now put $m=-n$ where n is +ive

$$\therefore\ f(-n).f(n)=f(-n+n)=f(0)=1$$

$$\therefore\ f(-n)=\frac{1}{f(n)}=\frac{1}{(1+x)^n}=(1+x)^{-n}.\ (n \text{ being +ive})$$

$$\text{Now } f(-n)=1+(-n)x+\frac{(-n)(-n-1)}{2\,!}x^2+\ldots\ldots$$

$$\text{or } (1+x)^{-n}=1-nx+\frac{n(n+1)}{2\,!}x^2-\ldots\ldots(2)$$

which proves binomial theorem for any negative index.

Note. Putting $x=-x$ we get

$$(1-x)^{-n}=1+nx+\frac{n(n+1)}{2\,!}x^2+\frac{n(n+1)(n+2)}{3\,!}x^3+\ldots\ldots(3)$$

It is easy to observe that in the case when n is not positive integral, then number of terms in the expansion of $(1+x)^{-n}$ is infinite

Putting $n=1$ in (3) we get.

$$(1-x)^{-1}=1+x+x^2+x^3+\ldots\ldots(4)$$

If $x=2$ we get from (4)

$$-1=1+2+2^2+2^3+\ldots\ldots \text{which is clearly absurd.}$$

If $x=\frac{1}{2}$ we get from (4)

$$\left(1-\frac{1}{2}\right)^{-1}=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots$$

$$\text{or}\quad \left(\frac{1}{2}\right)^{-1}=\frac{1}{1-\frac{1}{2}}\ \text{Summing an infinite G. P.}$$

or $2=2$ which is true.

Hence the expansion of $(1\pm x)^n$ when n is any index other than +ive integer is true only when x is less than unity.

It is easy to observe that T_{r+1} in the expansion of $(1-x)^{-n}$ is $=\dfrac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r}x^r$

Also if $(x+a)^n$ is to be expanded in the case when n is not a +ive integer, it should always be put in the form $(1\pm x)^n$.

Thus $(x+a)^n$ may be written $x^n(1+a/x)^n$ and its expansion will be valid only when $a/x < 1$ i.e. $a < x$ or $x > a$

Or $(x+a)^n$ may be put as $a^n(1+x/a)^n$ and its expansion will be valid only when $x/a < 1$ i.e. $x < a$.

§5. Some important expansions.

(i) $(1-x)^{-1}=1+x+x^2+x^3+\ldots\ldots x^r+\ldots\ldots$

$(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots\ldots$

$=1+(-1)^1x+(-1)^2x^2+(-1)^3x^3+\ldots+(-1)^rx^r+\ldots$

(ii) $(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots+(r+1)x^r+\ldots$

(iii) $(1+x)^{-2}=1-2x+3x^2+4x^3+\ldots(-1)^r(r+1)x^r+\ldots$

(iv) $(1-x)^{-3}=1+3x+6x^2+10x^3+\ldots+\frac{(r+1)(r+2)}{1.2}x^r+\ldots\ldots$

(v) $(1+x)^{-3}=1-3x+6x^2-10x^3+\ldots\ldots+(-1)^r\frac{(r+1)(r+2)}{1.2}x^r+\ldots\ldots$

Summation of series.

Ex. 1. Identifying as binomial expansion, show that $\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots\ldots=0{\cdot}4$ **nearly.**

(Rajputana 49)

We shall try to bring the above series in the form of the expansion of $(1-x)^{-n}$ i.e. $1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}\times x^3+\ldots\ldots$

$$\text{L. H. S.}=\frac{1.3}{1.2}\cdot\frac{1}{3^2}+\frac{1.3.5}{1.2.3}\cdot\frac{1}{3^3}+\frac{1.3.5.7}{1.2.3.4}\cdot\frac{1}{3^4}+\ldots\ldots$$

$$+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\cdot\frac{2^2}{3^2}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\cdot\frac{2^3}{3^3}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{4!}\cdot\frac{2^4}{3^4}+.$$

Put $x=\frac{2}{3}$ and $n=\frac{1}{2}$.

$$=\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\frac{n(n+1)(n+2)(n+3)}{4!}x^4+\ldots$$

$$=(1-x)^{-n}-(1+nx)=(1-\tfrac{2}{3})^{-\frac{1}{2}}-(1+\tfrac{2}{3}\cdot\tfrac{1}{2})$$

$=(\frac{1}{3})^{-\frac{1}{2}}-\frac{4}{3}=\sqrt{3}-\frac{4}{3}=1{\cdot}732-1{\cdot}33$ nearly $=0{\cdot}4$ nearly.

Ex. 2. If $y=\frac{1}{3}+\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}+\ldots\ldots\ldots$

prove that $y^2+2y-2=0$. **(Rajputana 54)**

From Ex. 1, $y=(1-x)^{-n}-1=(1-\frac{2}{3})^{-\frac{1}{2}}-1$

or $(y+1)=(\frac{1}{3})^{-\frac{1}{2}}=3^{\frac{1}{2}}$. Square both sides.

$$y^2+2y+1=3 \text{ or } y^2+2y-2=0.$$

Ex. 3. Sum the series to infinity

$$2+\frac{5}{2!\cdot 3}+\frac{5\cdot 7}{3!\cdot 3^2}+\frac{5\cdot 7\cdot 9}{4!\cdot 3^3}+\ldots\ldots$$

(Rajputana 52, Allahabad 46)

$$\text{L. H. S.}=2+\frac{\frac{5}{2}}{2!}\cdot\frac{2}{3}+\frac{\frac{5}{2}\cdot\frac{7}{2}}{3!}\cdot\left(\frac{2}{3}\right)^2+\frac{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{4!}\cdot\left(\frac{2}{3}\right)^3+\ldots\ldots\ldots$$

$$=2+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\cdot\left(\frac{2}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{3!}\cdot\left(\frac{2}{3}\right)^3$$

$$+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{4!}\cdot\left(\frac{2}{3}\right)^4+\ldots\ldots$$

Put $x=\frac{2}{3}$ and $n=\frac{3}{2}$.

$$\therefore\ \text{L. H. S}=2+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\ldots\ldots$$

$$=2+[(1-x)^{-n}-(1+nx)]$$

$$=2+[1-\tfrac{2}{3})^{-3/2}-(1+\tfrac{2}{3}\cdot\tfrac{3}{2})]$$

$$=2+[(\tfrac{1}{3})^{-3/2}-2]=(3)^{3/2}=(3^3)^{\frac{1}{2}}=\sqrt{(27)}$$

Ex. 4. **Prove that** $1=\frac{1}{4}+\frac{1\,.\,3}{4\,.\,6}+\frac{1\,.\,3\,.\,5}{4\,.\,6\,.\,8}+\ldots\ldots\infty$

(Agra 41)

$$\text{R. H. S.}=\frac{1}{4}+\frac{1\,.\,3}{2\,.\,3}\,.\,\frac{1}{2^2}+\frac{1\,.\,3\,.\,5}{2\,.\,3\,.\,4}\,.\,\frac{1}{2^3}+\ldots\ldots$$

$$=\frac{1}{4}+\frac{\frac{1}{2}\,.\,\frac{3}{2}}{3!}\,.\,1^2+\frac{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}}{4!}\,.\,1^3+\ldots\ldots$$

$$=\frac{\frac{1}{2}}{2!}\,.\,1+\frac{\frac{1}{2}\,.\,\frac{3}{2}}{3!}\,.\,1^2+\frac{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}}{4!}\,.\,1^3+\ldots\ldots$$

$$=2\left[\frac{\frac{1}{2}\,.\,\frac{1}{2}}{2!}\,.\,1^2+\frac{\frac{1}{2}\,.\,\frac{1}{2}\,.\,\frac{3}{2}}{3!}1^3+\frac{\frac{1}{2}\,.\,\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}}{4!}1^4+\ldots\right]$$

$$=-2\left[\frac{\frac{1}{2}(-\frac{1}{2})}{2!}\,.\,1^2-\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}\,.\,1^3+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}1^4-\ldots\ldots\right]$$

$$=-2\left[\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}1^2-\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}1^3+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}1^4-\ldots\right]$$

$=-2[(1-x)^{\frac{1}{2}}-(1-\frac{1}{2}x)]$ where $x=1$

$=-2[(1-1)^{\frac{1}{2}}-(1-\frac{1}{2})]=-2[0-\frac{1}{2}]=1$ **Proved**

Ex. 5. **Prove that** $1+\frac{2n}{3}+\frac{2n(2n+2)}{3\,.\,6}+\frac{2n(2n+2)(2n+4)}{3\,.\,6\,.\,9}+\ldots\ldots$

$$=2^n\left[1+\frac{n}{3}+\frac{n(n+1)}{3\,.\,6}+\frac{n(n+1)(n+2)}{3\,.\,6\,.\,9}+\ldots\ldots\right]$$

(Rangoon 38)

L. H. S. $=(1-\frac{2}{3})^{-n}=(\frac{1}{3})^{-n}=3^n$

R. H. S. $=2^n[(1-\frac{1}{3})^{-n}]=2^n\,.\,(\frac{2}{3})^{-n}=2^n(\frac{3}{2})^n=3^n$

Exercise

(1) $\sqrt{8}=1+\frac{3}{4}+\frac{3\,.\,5}{4\,.\,8}+\frac{3\,.\,5\,.\,7}{4\,.\,8\,.\,12}+\ldots$ (Calcutta 1894)

(2) Prove that $7^n\left\{1+\frac{n}{7}+\frac{n(n-1)}{7\,.\,14}+\frac{n(n-1)(n-2)}{7\,.\,14\,.\,21}+\ldots\right\}$

$$=4^n\left\{1+\frac{n}{2}+\frac{n(n+1)}{2\,.\,4}+\frac{n(n+1)(n+2)}{2\,.\,4\,.\,6}+\ldots\ldots\right\}$$

(3) $1\quad\frac{3}{4}+\frac{3\,.\,5}{4\,.\,8}+\frac{3\,.\,5\,.\,7}{4\,.\,8\,.\,12}+\ldots\ldots=(\frac{3}{2})^{-3/2}$

(Madras 48)

(4) $1-\frac{1}{8}+\frac{1\,.\,3}{8\,.\,16}+\frac{1\,.\,3\,.\,7}{8\,.\,16\,.\,24}+\ldots=\frac{2}{\sqrt{5}}$

Hint :—L. H. S. $=(1+\frac{1}{4})^{-1/2}$ etc.

(5) $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\,.\,5}{3\,.\,6}\cdot\frac{1}{2^2}+\frac{2\,.\,5\,.\,8}{3\,.\,6\,.\,9}\cdot\frac{1}{2^3}+\ldots=(4)^{1/3}$

(Utkal 49)

(6) $1+\frac{5}{8}+\frac{5\,.\,8}{8\,.\,12}+\frac{5\,.\,8\,.\,11}{8\,.\,12\,.\,16}+\ldots\ldots4(2)^{1/3}-2$

(Annamalai 49)

CHAPTER VII

EXPONENTIAL THEOREM

and

LOGARITHMIC SERIES

§1. What do e and e^x stand for :—Prove that e lies between 2 and 3 and that it is irrational (incommensurable)

$$e=\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^n \text{ and } e^x=\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^{nx}$$

We know from Binomial Theorem that if $1/n$ be <1 then

$$\left(1+\frac{1}{n}\right)^n=1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2\,!}\cdot\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3\,!}\left(\frac{1}{n}\right)^3+\dots\dots\dots$$

$$=1+1+\frac{\left(1-\frac{1}{n}\right)}{2\,!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3\,!}+\dots\dots \text{I}$$

Now the above relation holds good for all values of n choosing n very large so that products of the form

$\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)$ etc. all tend to 1 as $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$ will tend to zero. Hence taking limit of both sides of 1 when n is infinite we get

$$e=\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\dots\infty$$

$$=1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\dots\infty$$

Similarly proceeding as above we can prove that

$$e^x = \underset{n\to\infty}{\mathrm{Lt}} \left(1+\frac{1}{n}\right)^{nx} \text{ and also} = \left(1+\frac{x}{n}\right)^n$$

$$=1+\frac{x}{1\,!}+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\frac{x^4}{4\,!}+\ldots\ldots\infty \quad (2)$$

e lies between 2 & 3

$$e=1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\ldots\ldots\ldots\infty \quad (3)$$

clearly e is greater than 2 and, since

$$\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\ldots\ldots \textit{i.e.}\ \frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots\ldots$$

is less than $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\ldots$

or less than $\frac{\frac{1}{2}}{1-\frac{1}{2}}$ (Summing an infinite G. P.) or <1,

hence e is less than $1+\frac{1}{1\,!}+1$ *i.e.* less than 3 from (3)

Therefore e is greater than 2 but less than 3.

e is irrational.

If possible suppose $e=p/q$ where p & q are +ive integers, then

$$e=\frac{p}{q}=\left(1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots\right.$$

$$\left.+\frac{1}{q\,!}\right)+\frac{1}{(q+1)\,!}+\frac{1}{(q+2)\,!}+\ldots\ldots$$

Now multiply both sides by $q\,!$ we get

$$\frac{p}{q}\,q\,!=\left(1+\frac{1}{1\,!}+\frac{1}{2\,!}+\ldots\ldots\frac{1}{q\,!}\right)q\,!+\frac{q\,!}{(q+1)\,q\,!}$$

$$+\frac{q\,!}{(q+2)(q+1)\,q\,!}+\ldots\ldots$$

$$\text{or } p\,.\,(q-1)!=\text{An integer}+\frac{1}{q+1}+\frac{1}{(q+2)(q+1)}+\ldots\ldots(4)$$

Now $\frac{1}{q+1}+\frac{1}{(q+2)(q+1)}+\ldots\ldots$

is less than $\frac{1}{q+1}+\frac{1}{(q+1)^2}+\ldots\ldots\infty$

or less then $\dfrac{\frac{1}{q+1}}{1-\frac{1}{q+1}}$ [Summing a G P.] or less than $\frac{1}{q}$

Also $\frac{1}{q+1}+\frac{1}{(q+2)(q+1)}+\ldots\ldots$ is clearly greater than $\frac{1}{q+1}$ Thus $\frac{1}{q+1}+\frac{1}{(q+2)(q+1)}+\ldots\ldots$ is a fraction whose sum lies between $\frac{1}{q}$ & $\frac{1}{q+1}$ and is thus a proper fraction. Hence from (4) we get

An integer = an integer + fraction, which is clearly absurd. Therefore e is irrational.

§2. Some important results

(References are made to the following results in the solutions)

1. $e=\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots\infty$

and it lies between 2 and 3.

2. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\ldots\infty$

3. $e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\ldots\ldots\infty$

4. $\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\infty$

5. $\frac{e^x-e^{-x}}{2!}=\frac{x}{1!}+\frac{x^3}{3!}+\ldots\ldots\infty$

Putting $x=1$ in (2), (3), (4) and (5) we find that

6. $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\infty$

7. $e^{-1}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\ldots\infty$

or $=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\ldots\infty$

8. $\frac{e^1+e^{-1}}{2}=1+\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots\infty$

9. $\frac{e^1-e^{-1}}{2}=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots\infty$

***Ex. 1. Find the sum to infinity of the series**

$$1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\ldots\ldots\infty$$

(Agra 46, 50, Utkal 48)

$$T_n=\frac{n^3}{n!}=\frac{n^2}{(n-1)!}=\frac{(n-1)(n-2)+3n-2}{(n-1)!}$$

or $T_n=\frac{1}{(n-3)!}+\frac{3n-3+1}{(n-1)!}=\frac{1}{(n-3)!}+3\cdot\frac{1}{(n-2)!}+\frac{1}{(n-1)!}$

Putting $n=1, 2, 3, 4\ldots\ldots$ and remembering that

$$\frac{1}{(-n)!}=0 \text{ and } 0!=1$$

$T_1=0+3\cdot 0+1$

$T_2=0+3\cdot 1+\frac{1}{1!}$

$T_3=1+3\cdot\frac{1}{1!}+\frac{1}{2!}$

$T_4=\frac{1}{1!}+3\cdot\frac{1}{2!}+\frac{1}{3!}$ and so on

Adding vertically we get

$$S\infty=\left(1+\frac{1}{1!}+\ldots\ldots\right)+3\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots\right)$$

$$+\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\right)$$

$$=e+3e+e=5e$$

Ex. 2. **Show that** $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\ldots\ldots=e^{-1}$

(Agra 43, Rangoon 37)

$$T_n=\frac{2n}{(2n+1)!}=\frac{2n+1-1}{(2n+1)!}=\frac{1}{(2n)!}-\frac{1}{(2n+1)!}$$

$$\therefore\quad T_1=\frac{1}{2!}-\frac{1}{3!}$$

$$T_2=\frac{1}{4!}-\frac{1}{5!}$$ and so on

$$\therefore\quad S\infty=\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots\infty\right)=e^{-1}$$ **(Result 7)**

Ex. 3. **Show that** $3+\frac{5}{1!}+\frac{7}{2!}+\frac{9}{3!}+\ldots\ldots=5e$

(Patna 38)

$$T_n=\frac{2n+1}{(n-1)!}=\frac{2(n-1)+3}{(n-1)!}$$

or $T_n=\frac{2}{(n-2)!}+\frac{3}{(n-1)!}$ etc. $\therefore\quad S\infty=5e$

Ex. 4. $\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots\right)^2=1+\left(1+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots\right)^2$

L. H. S. $=\left(\frac{e+e^{-1}}{2}\right)^2=\frac{e^2+e^{-2}+2e\,.\,e^{-1}}{4}=\frac{e^2+e^{-2}+2}{4}$

Result 8

R. H. S. $=1+\left(\frac{e-e^{-1}}{2}\right)^2=\frac{4+(e^2+e^{-2}-2e^1\,.\,e^{-1})}{4}$

$=\frac{e^2+e^{-2}+2}{4}=$ L. H. S.

Ex. 5. Sum the series

$$\frac{1}{1!}+\frac{1+3}{2!}x+\frac{1+3+5}{3!}x^2+\frac{1+3+5+7}{4!}x^3+\ldots\infty$$

(Nagpur 40)

$$T_n=\frac{1+3+5+7+\ldots\ldots n\text{ terms}}{n!}x^{n-1}=\frac{n}{2}\,\frac{[2.1+(n-12)}{n!}x^{n-1}]$$

$$=\frac{n^2}{n!}\,.\,x^{n-1}=\frac{n}{(n-1)!}x^{n-1}=\frac{n-1+1}{(n-1)!}x^{n-1}$$

or $T_n=\frac{1}{(n-2)!}\,.\,x^{n-1}+\frac{1}{(n-1)!}\,.\,x^{n-1}$.

$T_1=0+1$

$T_2=x+\frac{x}{1!}$

$T_3=\frac{x^2}{1!}+\frac{x^2}{2!}$

$T_4=\frac{x^3}{2!}+\frac{x^3}{3!}$ and so on

$$\therefore\ S_\infty=\left(x+\frac{x^2}{1!}+\frac{x^3}{2!}+\ldots\right)+\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\infty\right)$$

$$=x\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\right)+\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\infty\right)$$

$=xe^x+e^x=(x+1)e^x$

Ex. 6. Show that

$$1+\frac{1+a}{2!}+\frac{1+a+a^2}{3!}+\frac{1+a+a^2+a^3}{4!}+\ldots\ldots=\frac{e^a-e}{a-1}$$

$$T_n=\frac{1+a+a^2+a^3+a^4+\ldots\ldots\ldots n \text{ terms}}{n!}=\frac{1\,.\,(a^n-1)}{(a-1)\,.\,n!}$$

$$=\frac{1}{(a-1)}\left[\frac{a^n}{n!}-\frac{1}{n!}\right] \quad \text{Putting } n=1,\,2,\,3\ldots\ldots$$

$$T_1=\frac{1}{(a-1)}\left[\frac{a}{1!}-\frac{1}{1!}\right]$$

$$T_2=\frac{1}{(a-1)}\left[\frac{a^2}{2!}-\frac{1}{2!}\right]$$

$$T_3=\frac{1}{(a-1)}\left[\frac{a^3}{3!}-\frac{1}{3!}\right] \text{ and so on}$$

$$\therefore\ S_\infty=\frac{1}{(a-1)}\left[\left\{\frac{a}{1!}+\frac{a^2}{2!}+\frac{a^3}{3!}\ldots\ldots\right\}-\left\{\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\right\}\right]$$

$$=\frac{1}{(a-1)}[\{(e^a-1)-(e^1-1))\}] \quad \text{(Note)}$$

$$=\frac{e^a-e}{a-1}$$

Note. If we put $a=2$, we get

$$1+\frac{1+2}{2!}+\frac{1+2+2^2}{3!}+\frac{1+2+2^2+2^3}{4}+\ldots\ldots\infty=e^2-$$

(Travancore 46, Rangoon 39, 50)

Ex. 7. Prove that $1+\frac{2^4}{2!}+\frac{3^4}{3!}+\frac{4^4}{4!}+\ldots\ldots=15e$

(Patna 37, Madras 40)

$$T_n=\frac{n^4}{n!}=\frac{n^3}{(n-1)!}=\frac{(n-1)(n-2)(n-3)+6n^2-11n+6}{(n-1)!}$$

$$=\frac{1}{(n-4)!}+\frac{6n^2-11n+6}{(n-1)!}$$

$$=\frac{1}{(n-4)!}+\frac{6.(n-1)(n-2)+18n-12-11n+6}{(n-1)!}$$

$$=\frac{1}{(n-4)!}+\frac{6}{(n-3)!}+\frac{7n-6}{(n-1)!}$$

$$=\frac{1}{(n-4)!}+\frac{6}{(n-3)!}+\frac{7(n-1)+7-6}{(n-1)!}$$

$$=\frac{1}{(n-4)!}+\frac{6}{(n-3)!}+\frac{7}{(n-2)!}+\frac{1}{(n-1)!}$$

Now putting $n=1, 2, 3$......and so on and adding we get

$S_\infty=e+6e+7e+e=15e$

Ex. **Prove that** $1+\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+......\infty$

(Bombay 48)

$=(x^3+6x^2+7x+1)e^x$

Let $S=\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+\frac{4^3}{3!}x^3+......\infty$

and hence we have to find the value of $1+S$

$$T_n=\frac{(n+1)^3}{n!}x^n=\left(\frac{n^3+3n^2+3n+1}{n!}\right)x^n$$

$$=\frac{n^2+3n+3}{(n-1)!}x^n+\frac{1}{n!}x^n$$

$$=\frac{(n-1)(n-2)+6n+1}{(n-1)!}x^n+\frac{1}{n!}x^n$$

$$=\frac{1}{(n-3)!}x^n+\frac{6.(n-1)+7}{(n-1)!}x^n+\frac{1}{n!}x^n$$

$$=x^3.\frac{x^{n-3}}{(n-3)!}+\frac{6}{(n-2)!}x^n+\frac{7}{(n-1)!}x^n+\frac{1}{n!}x^n$$

$$=x^3.\frac{x^{n-3}}{(n-3)!}+6.x^2.\frac{x^{n-2}}{(n-2)!}+7x.\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}$$

Putting $n=1, 2, 3$......and then adding we get

$S=x^3 . e^x+6x^2 . e^x+7x . e^x+(e^x-1)$ (Note)

$\therefore\ 1+S=(x^3+6x^2+7x+1)e^x$

Ex. 9. By comparing two different expansions of $\left(\frac{e^x-e^{-x}}{2}\right)^n$ show that $n^n-n(n-2)^n+\frac{n(n-1)}{2!}(n-4)^n-$...... ...to $(n+1)$ terms $=2^n . n!$

$$\left(\frac{e^x-e^{-x}}{2}\right)^n=\frac{1}{2^n}\left[e^{nx}-n\ e^{(n-1)x} . e^{-x}+\frac{n(n-1)}{2!}\right.$$

$$\left. e^{(n-2)x} . e^{-2x}-\ldots\ldots\ldots\right]$$

$$=\frac{1}{2^n}\left[e^{nx}-n\ e^{(n-2)x}+\frac{n(n-1)}{2!}e^{(n-4)x}-\ldots\ldots\right]\ \ldots A$$

Now $e^{nx}=1+(nx)+\frac{(nx)^2}{2!}+\frac{(nx)^3}{3!}+\ldots\ldots+\frac{(nx)^n}{n!}+\ldots\ldots$

i.e the coefficient of x^n in the expansion of e^{nx} is $\frac{n^n}{n!}$

Similarly ,, ,, ,, ,, $-ne^{(n-2)x}$ is $-\frac{n(n-2)^n}{n!}$

,, ,, ,, ,, ,, $\frac{n(n-1)}{2!}e^{(n-4)x}$ is

$\frac{n(n-1)}{2!} . \frac{(n-4)^n}{n!}$ ......and so on

Hence the coefficient of x^n in A is

$$\frac{1}{2^n}\left[\frac{n^n}{n!}-\frac{n(n-2)^n}{n!}+\frac{n(n-1)}{2!} . \frac{(n-4)^n}{n!}-\ldots(n+1)\text{ terms}\right]$$

Again $\left(\frac{e^x-e^{-x}}{2}\right)^n=\left(\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\ldots\right)^n$ (Result 5)

$$=x^n\left[1+\left(\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots\ldots\ldots\right)\right]^n$$

and clearly the coefficient of x^n is unity.

Equating the coefficients of x^n obtained from R. H. S. and L. H. S. of (A) we get the required result.

Ex. 10. By means of the identity

$$e^{x^2+\frac{1}{x^2}+2}=e^{\left(x+\frac{1}{x}\right)^2}$$

or otherwise, show that

$$e^2\left\{1+\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\frac{1}{(3!)^2}+\dots\dots\right\}$$

$$=1+\frac{2!}{(1!)^3}+\frac{4!}{(2!)^3}+\frac{6!}{(3!)^3}+\dots\dots$$

$$e^{x^2+\frac{1}{x^2}+2}=e^{x^2}\cdot e^{\frac{1}{x^2}}\cdot e^2$$

$$\left\{1+\frac{(x^2)}{1!}+\frac{(x^2)^2}{2!}+\frac{(x^2)^3}{3!}+\dots\dots\right\}$$

$$\left\{1+\frac{1}{1!}\cdot\left(\frac{1}{x^2}\right)+\frac{1}{2!}\left(\frac{1}{x^2}\right)^2+\frac{1}{3!}\left(\frac{1}{x^2}\right)^3+\dots\dots\right\}$$

Clearly on multiplication the term independent of x is

$$=e^2\left\{1+\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\frac{1}{(3!)^2}+\dots\dots\right\}\dots\dots(A)$$

Again $e^{\left(x+\frac{1}{x}\right)^2}=\left\{1+\frac{\left(x+\frac{1}{x}\right)^2}{1!}+\frac{\left(x+\frac{1}{x}\right)^4}{2!}\right.$

$$\left.+\frac{\left(x+\frac{1}{x}\right)^6}{3!}+\dots\dots\right\}$$

Now $\left(x+\frac{1}{x}\right)^{2n}$ contains $(2n+1)$ terms and its middle i. e. $(n+1)$th will clearly be independent of x.

$$T_{n+1}={}^{2n}C_n\, x^{2n-n}\cdot\frac{1}{x^n}={}^{2n}C_n=\frac{2n!}{n!\,.\,(2n-n)\,!}=\frac{2n!}{(n!)^2}$$

Putting $n=1$, 2, 3 and so on the term independent of x

in the expansions of $\left(x+\frac{1}{x}\right)^2$, $\left(x+\frac{1}{x}\right)^4$, $\left(x+\frac{1}{x}\right)^6$ and so on are respectively $\frac{2!}{(1!)^2}$, $\frac{4!}{(2!)^2}$, $\frac{6!}{(3!)^2}$ and so on.

Hence the term independent of x^n in the expansion of

$$e^{\left(x+\frac{1}{x}\right)^2} \text{ is } \left[1+\frac{1}{1!}\cdot\frac{2!}{(1!)^2}+\frac{1}{2!}\cdot\frac{4!}{(2!)^2}+\frac{1}{3!}\cdot\frac{6!}{(2!)^2}+\ldots\ldots\right] \qquad \ldots\ldots(B)$$

Equating (A) and (B) we get the required result.

Exercise

1. Expand $(e^{5x}+e^{x})/e^{3x}$ in ascending powers of x.

Ans. $2\left\{1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\ldots\ldots\right\}$ (Result 4)

2. $\frac{1^2}{2!}+\frac{2^2}{3!}+\frac{3^2}{4!}+\ldots\ldots\infty=e-1$

3. $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots\ldots=\frac{3e}{2}$

(Rangoon 48, Poona 50)

4. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\ldots=2e$

5. $\frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{37}{4!}+\frac{56}{5!}+\ldots\ldots=6e-1$

Hint. 4, 11. 22, 37, 56...... are such that the successive differences 7. 11, 15, 19 etc. are in A. P. and hence

$$T_n=\frac{2n^2+n+1}{n!} \text{ etc.}$$

6. Prove that the coefficient of x^n in the expansion of

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\right)^2 \text{ is } \frac{2^{n-1}}{n!}$$

Hint. $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\right)^2=\left(\frac{e^x+e^{-x}}{2}\right)^2$

$$=\tfrac{1}{4}(e^{2x}+e^{\ \ 2x}+2)$$

$$=\tfrac{1}{2}+\tfrac{1}{2}\frac{(e^{2x}+e^{-2x})}{2}=\tfrac{1}{2}+\tfrac{1}{2}\left\{1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\ldots\right\}$$ (Result 4)

(7) **Show that** $1+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\frac{1^2+2^2+3^2+4^2}{4!}$

$\ldots\ldots\ldots=\frac{17}{6}e$.

Hint $T_n=\frac{1^2+2^2+3^2+\ldots\ldots n^2}{n!}=\frac{n(n+1)(2n+1)}{6\,.\,n!}=$etc.

(8) Show that the coefficient of x^n in the series

$1+\frac{a+bx}{1!}+\frac{(a+bx)^2}{2!}+\ldots$ is $e^a\,.\,\frac{b^n}{n!}$ (U. P. C. S. 47)

(9) By comparing two different expansions of $\left(\frac{e^x+e^{-x}}{2}\right)^n$

prove that $1\,.\,n^2+n(n-2)^2+\frac{n(n-1)}{2!}(n-4)^2+\ldots(n+1)$

terms $=2^n\,.\,n$

Hint. See Ex. 9.

(10) By comparing two different expansions of $(e^x-1)^n$ prove that **(Sagar 50)**

$n^n-n(n-1)^n+\frac{n(n-1)}{2!}\,.\,(n-2)^n-\ldots\ldots$to n terms$=n!$

Hint. See Ex. 9.

(11) Show that $\left(n+\frac{1}{n}\right)=2\left\{1+\frac{(\log n)^2}{2!}+\frac{(\log n)^4}{4!}+\ldots\right\}$

(Patna 50)

Hint. Use Result (4) and remember that $e^{\log x}=x$ etc.

Logarithmic Series

Note :–The base e is understood unless stated otherwise

§3. Prove that if x < 1, then

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$$

Also find an expansion for $\{\log(1+x)\}^2$

(Agra 53, Bombay 28)

We know that $a^y = e^{y \log a}$

$$\text{or } a^y = 1 + (y \log a) + \frac{(y \log a)^2}{2\,!} + \frac{(y \log a)^3}{3\,!} + \ldots\ldots$$

Now put $a = (1+x)$ in both sides

$$\therefore\quad (1+x)^y = 1 + y \log(1+x) + \frac{y^2}{2\,!}\{\log(1+x)\}^2 + \frac{y^3}{3\,!}\cdot\{\log(1+x)\}^3 + \ldots\ldots$$

Now as x is less than unity we have by Binomial Theorem

$$1 + yx + \frac{y(y-1)}{2\,!}x^2 + \frac{y(y-1)(y-2)}{3\,!}x^3 + \frac{y(y-1)(y-2)(y-3)}{4\,!}x^4 + \ldots$$

$$= 1 + y \log(1+x) + \frac{y^2}{2\,!}[\log(1+x)]^2 + \frac{y^3}{3\,!}[\log(1+x)]^3 + \ldots$$

Equating the coefficient of y in both sides of above we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots$$

Again the coefficient of y^2 in the R. H. S. is $\frac{1}{2\,!}[\log(1+x)]^2$ and the coefficient of y^2 in L. H. S. is the coefficient of y in

$$\frac{(y-1)}{2\,!}x^2 + \frac{(y-1)(y-2)}{3\,!}x^3 + \frac{(y-1)(y-2)(y-3)}{4\,!}x^4 - \ldots$$

and is equal to $\frac{x^2}{1.2} - \frac{1+2}{3\,!}x^3 + \frac{1.2+2.3+3.1}{4\,!}x^4 - \ldots$

$$\therefore\quad \frac{1}{2\,!}[\log(1+x)]^2 = \frac{x^2}{1\;2} - \frac{1+2}{1.2.3}x^3 + \frac{1\,.\,2+2\,.\,3+3\,.\,1}{1\,.\,2\,.\,3\,.\,4}x^4 - \ldots$$

$$\text{or } [\log(1+x)]^2 = 2\left\{\frac{x^2}{2} - \frac{x^3}{3}\cdot\frac{(1+2)}{1.2} + \frac{x^4}{4}\cdot\frac{(1.2+2.3+3.1)}{1.2.3} - \ldots\ldots\right\}$$

$$=2\left\{\frac{x^2}{2}-(1+\tfrac{1}{2})\cdot\frac{x^3}{3}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{x^4}{4}-\ldots\ldots\right\}$$

§ 4. **Various results.**

(References are made to the following results in the solutions)

(1) $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots\ldots$

Putting $x=-x$ we get,

(2) $\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots\ldots$

or $-\log(1-x)=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\ldots\ldots\right)$

Subtracting 1 & 2 we get

(3) $\log(1+x)-\log(1-x)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\ldots\right)$

or $\frac{1}{2}\log\frac{1+x}{1-x}=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\ldots$

Putting $\frac{1+x}{1-x}=\frac{m}{n}$ $\therefore$ $x=\frac{m-n}{m+n}$ we get

(4) $\frac{1}{2}\cdot\log\frac{m}{n}=\left[\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\ldots\ldots\right]$

(Rangoon 50)

Ex. 1. Prove that $2\log n-\log(n+1)-\log(n-1)$

$=\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+\ldots\ldots$ **(Agra 48)**

Put $\frac{1}{n^2}=x$ $\therefore$ R.H.S.$=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots$

$=-\log(1-x)$ (by result 2)$=-\log\left(1-\frac{1}{n^2}\right)$

$=-\log\frac{(n^2-1)}{n^2}$

$=\log\frac{n^2}{n^2-1}=\log n^2-\log(n-1)(n+1)$

$=2\log n-\log(n-1)-\log(n+1)$

Ex. 2. Prove that $2\log n - \log(n+1) - \log(n-1)$

$$=2\left\{\frac{1}{(2n^2-1)}+\frac{1}{3(2n^2-1)^3}+\frac{1}{5(2n^2-1)^5}+\ldots\ldots\right\}$$

Put $\frac{1}{2n^2-1}=x \quad \therefore \quad \text{R.H.S.}=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots\right\}$

$$=\log\frac{1+x}{1-x}\ \text{(Result 3)}=\log\frac{1+\frac{1}{2n^2-1}}{1-\frac{1}{2n^2-1}}$$

$$=\log\frac{2n^2}{2n^2-2}=\log\frac{n^2}{n^2-1}=\log n^2-\log(n-1)(n+1)$$

$$=2\log n-\log(n-1)-\log(n+1)$$

Ex. 3, Prove that $\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\ldots\ldots$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots$$ (Agra 49)

Putting $x=\frac{1}{n+1}$, $\text{L.H.S.}=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots$

$$=-\log(1-x)\ \text{(Result 2)}=-\log\left(1-\frac{1}{n+1}\right)$$

$$=-\log\frac{n}{n+1}=\log\frac{n+1}{n}=\log\left(1+\frac{1}{n}\right)$$

$$=\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n}\right)^2+\frac{1}{3}\left(\frac{1}{n}\right)^3-\ldots\ldots\text{(Result 1)}$$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots$$

Ex. 4. If x is numerically less than unity, prove that

$$\tfrac{1}{2}x^2+\tfrac{2}{3}x^3+\tfrac{3}{4}x^4+\tfrac{4}{5}x^5+\ldots\ldots=\frac{x}{1-x}+\log(1-x)$$

(Agra 51)

L.H.S. $=(1-\frac{1}{2})x^2+(1-\frac{1}{3})x^3+(1-\frac{1}{4})x^4$
$+(1-\frac{1}{5})x^5+\dots\dots\dots\infty$

$$=\left(x^2+x^3+x^4+x^5+\dots\dots\right)-\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\dots\right).$$

Adding x in both the brackets we get

$$=(x+x^2+x^3+x^4+x^5+\dots\dots\dots)$$
$$-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\dots\dots\right)$$

$=\frac{x}{1-x}+\log(1-x)$ by (Result 2) and sum of a G. P.

Ex. 5. If $\log(1-x+x^2)$ **be expanded in ascending powers of** x **in the form** $a_1x+a_2x^2+a_3x^3+\dots\dots\dots$ **show that** $a_3+a_6+a_9+\dots\dots\dots$

$=\frac{2}{3}\log 2$ **(Bombay 26, Rajputana 50)**

$$\log(1-x+x^2)=\log\frac{1+x^3}{1+x}=\log(1+x^3)-\log(1+x)$$

$$=\left\{x^3-\tfrac{1}{2}x^6+\tfrac{1}{3}x^9-\tfrac{1}{4}x^{12}+(-1)^{n-1}\frac{x^{3n}}{n}+\dots\dots\right\}$$
$$-\left\{x-\frac{x^2}{2}+\frac{x^3}{x}-\dots\dots+(-1)^{n-1}.\frac{x^n}{n}-\dots\dots\right\}$$

$=a_1x+a_2x^2+a_3x^3+\dots\dots$given. Hence a_n is the coefficient of x^n. Hence a_{3n} is the coefficient of x^{3n} in the above expansion and is equal to

$$\frac{(-1)^{n-1}}{n}-(-1)^{3n-1}.\frac{1}{3n}=\frac{(-1)^{n-1}}{n}-\frac{(-1)^{2n}.(-1)^{n-1}}{3n}$$

or $a_{3n}=(-1)^{n-1}\left[\frac{1}{n}-\frac{1}{3n}\right]=(-1)^{n-1}.\frac{2}{3n}$

$\because (-1)^{2n}=1$ or a_{3n}

Putting $n=1, 2, 3\dots\dots$we get

$a_3=\frac{2}{3},\ a_6=-\frac{2}{3}.\frac{1}{2},\ a_9=\frac{2}{3}.\frac{1}{3},\ a_{12}=-\frac{2}{3}.\frac{1}{4}\dots\dots$

$\therefore a_3+a_6+a_9+\dots\dots=\frac{2}{3}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\dots)$
$=\frac{2}{3}\log(1+1)=\frac{2}{3}\log 2$

Ex. 6. If log $(1+x+x^2+x^3)$ be expanded in ascending powers of x, show that the coefficient of x^n is $1/n$ if n is odd or of the form $4m+2$ and $-3/n$ if n is of the form $4m$. (Agra 29)

$$\log(1+x+x^2+x^3)=\log(1+x)(1+x^2)=\log(1+x)+\log(1+x^2)$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots(-1)^{n-1}\frac{x^n}{n}+\ldots\ldots\right)$$

$$+\left[x^{\frac{1}{2}}-\tfrac{1}{2}(x^2)^2+\tfrac{1}{3}(x^2)^3-\tfrac{1}{4}(x^2)^4\right.$$

$$+\ldots\ldots+\ldots\ldots\frac{(-1)^{n/2-1}}{\frac{n}{2}}(x^2)^{n/2}+\ldots$$

Case I. When n is odd then 2nd, bracket shall not give us the term of x^n and as such from first bracket the coefficient of x^n is $\frac{(-1)^{n-1}}{n}$, but n being odd $\therefore$ $(n-1)$ is even as such $(-1)^{n-1}$ is $+1$. Therefore the required coefficient is $1/n$.

Case II. When n is even$=4m+2$.

$\therefore$ $\frac{n}{2}=2m+1=$odd. Then the coefficient of x^n from second bracket is

$$\frac{(-1)^{\frac{n}{2}-1}}{n/2}=\frac{2(-1)^{\text{odd}-1}}{n}=\frac{2(-1)^{\text{even}}}{n}=2/n$$

Hence the total coefficient is $-\frac{1}{n}+\frac{2}{n}=\frac{1}{n}$

$-1/n$ being the coefficient from the first bracket.

Case III. When n is even$=4m$ $\therefore$ $n/2=2m=$even and proceeding as above in Case II the required coefficient is

$$-\frac{1}{n}-\frac{2}{n}=-\frac{3}{n}$$

Ex. 7 Prove that $1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}$

$$+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\dots=\frac{1}{2}\log 12$$

(Rajputana 51, 53)

$$\text{L.H.S.}=\left[1+\frac{1}{3}\cdot\frac{1}{2^2}+\frac{1}{5}\cdot\frac{1}{2^4}+\frac{1}{7}\cdot\frac{1}{2^6}+\dots\dots\right]$$

$$+\left[\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{4}\cdot\frac{1}{2^4}+\frac{1}{6}\cdot\frac{1}{2^6}\right]$$

$$=2\left[\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}+\frac{1}{7}\cdot\frac{1}{2^7}+\dots\dots\right]$$

$$+\left[\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{4}\cdot\frac{1}{2^4}+\frac{1}{6}\cdot\frac{1}{2^6}+\dots\right]$$

Let $x=\frac{1}{2}$ $\therefore$ $\text{L.HS.}=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\dots\right]$

$$+\left[\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots\dots\right]$$

$$=\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\dots\right]+\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\dots\right]$$

$$=\frac{1}{2}\log\frac{1+x}{1-x}-\log(1-x)=\frac{1}{2}\log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}-\log(1-\tfrac{1}{2})=\frac{1}{2}\log 3$$

$$-\log\tfrac{1}{2}$$

$=\frac{1}{2}\log 3+\log 2$ $\because$ $\log 1/x=\log x^{-1}=-\log x$

$=\frac{1}{2}[\log 3+2\log 2]=\frac{1}{2}[\log 3+\log 2^2]$ $\because$ $\log m^n=n\log m$

$=\frac{1}{2}[\log 3+\log 4]=\frac{1}{2}\log 3\times 4=\frac{1}{2}\log 12$ $\because$ $\log m$

$+\log n=\log mn$

Ex. 8. If α and β are the roots of $ax^2+bx+c=0$ show that

$$\log(a-bx+cx^2)=\log a+(\alpha+\beta)x-\frac{(\alpha^2+\beta^2)}{2}x^2$$

$$+\frac{(\alpha^3+\beta^3)}{3}x^3-\dots\dots$$

Now α and β are given to be the roots of $ax^2+bx+c=0$

$\therefore$ $\alpha+\beta=-b/a$ and $\alpha\beta=c/a$.

$$\therefore\ \log(a-bx+cx^2)=\log a[1-(b/a)x+(c/a)x^2]$$

$$=\log a[1+(\alpha+\beta)x+\alpha\beta x^2]=\log a(1+\alpha x)(1+\beta x)$$

$$=\log a+\log(1+\alpha x)+\log(1+\beta x)$$

$$=\log a+\left\{\alpha x-\frac{(\alpha x)^2}{2}+\frac{(\alpha x)^3}{3}-\frac{(\alpha x)^4}{4}+\ldots\ldots\right\}$$

$$+\left\{\beta x-\frac{(\beta x)^2}{2}+\frac{(\beta x)^3}{3}-\frac{(\beta x)^4}{4}+\ldots\right\}$$

$$=\log a+(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3-\ldots\ldots$$

Ex. 9. $\log\left(1+\frac{1}{n}\right)^n=1-\frac{1}{2(n+1)}-\frac{1}{2.3(n+1)^2}$

$$-\frac{1}{3.4(n+1)^3}-\ldots\ldots;$$ **(Rangoon 39)**

$$\text{R.H.S.}=1-\left(1-\frac{1}{2}\right)\cdot\frac{1}{(n+1)}-\left(\frac{1}{2}-\frac{1}{3}\right)\cdot\frac{1}{(n+1)^2}$$

$$-\left(\frac{1}{3}-\frac{1}{4}\right)\cdot\frac{1}{(n+1)^3}-\ldots$$

$$=-\left\{\frac{1}{(n+1)}+\frac{1}{2}\cdot\frac{1}{(n-1)^2}+\frac{1}{3}\cdot\frac{1}{(n+1)^3}-\ldots\ldots\right\}$$

$$+\left\{1+\frac{1}{2}\cdot\frac{1}{(n+1)}+\frac{1}{3}\cdot\frac{1}{(n+1)^2}+\ldots\ldots\right\}$$

Now put $\frac{1}{n+1}=x$

$$-\left\{x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\right\}+\frac{1}{x}\left\{x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\right\}$$

$$=-\left\{x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\right\}\left\{1-\frac{1}{x}\right\}=\left(1-\frac{1}{x}\right)\log(1-x)$$

Substituting the value of x we get

$$=\Big[1-(n+1)\Big]\left[\log\left(1-\frac{1}{n+1}\right)\right]=-n.\log\frac{n}{n+1}$$

$$=n\log\frac{n+1}{n}$$

$$=n\log(1+1/n)=\log(1+1/n)^n$$

Ex. 10. By means of the expansion of e^x and log $(1+x)$ prove that when n is large, (Agra 1954)

$$\left(1+\frac{1}{n}\right)^n=e\left(1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{17}{16n^3}+\dots\dots\right)$$

Let $K=\left(1+\frac{1}{n}\right)^n$ $\therefore$ $\log K=n\log\left(1+\frac{1}{n}\right)$

or $\log K=n\left\{\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\dots\dots\right\}$

$$=1-\left(\frac{1}{2n}-\frac{1}{3n^4}+\frac{1}{4n^3}-\dots\dots\right)$$

$$\therefore\quad K=e^{1-\left(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\dots\dots\right)}$$

$$=e\,.\,e^{-\left(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\dots\dots\right)}$$

$$=e\left[1-\left\{\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\dots\dots\right\}+\frac{1}{2\,!}\left\{\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\dots\right\}^2-\frac{1}{3\,!}\left\{\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\dots\right\}^3-\dots\right]$$

$$=e\left[1-\frac{1}{2n}+\frac{1}{n^2}\left(\frac{1}{3}+\frac{1}{8}\right)-\frac{1}{n^3}\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{48}\right)+\dots\dots\right]$$

$$=e\left[1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{17}{16n^3}+\dots\dots\right]$$

Exercise

1. Show that $\log 10=3\log 2+\frac{1}{4}-\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{3}\left(\frac{1}{4}\right)^3-\dots\dots$ (Madras 49)

2. Expand $\log\frac{1}{1-x-x^2+x^3}$ (Madras 48)

$$3x-\frac{5x^2}{2}+9\frac{x^3}{3}-17\frac{x^4}{4}+\dots\dots+(-1)^{r-1}(2^r-1)\frac{x^r}{r}+\dots$$

3. $\log x = \frac{x-1}{x+1} + \frac{1}{2} \cdot \frac{x^2-1}{(x+1)^2} + \frac{1}{3} \cdot \frac{x^3-1}{(x+1)^3} + \ldots\ldots$

(Rangoon 47, Poona 49)

Hint. Split into two series

4. $\log\left(\frac{x+1}{x}\right) = 2\left(\frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \ldots\right)$

(Travancore 45, Utkal 46)

5. $\log(x+n) = \log x + \log\left(1+\frac{1}{x}\right) + \log\left(1+\frac{1}{1+x}\right)$

$+\log\left(1+\frac{1}{2+x}\right) + \ldots. + \log\left(1+\frac{1}{n-1+x}\right)$

(Rangoon 38)

Hint. Use $\log m + \log n + \log p + \ldots\ldots = \log m \cdot n \cdot p \ldots$

6. If α and β are the roots of $x^2 - px + q = 0$, show that

$\log(1+px+qx^2) = (\alpha+\beta)x - (\alpha^2+\beta^2)\frac{x^2}{2} + (\alpha^3+\beta^3)\frac{x^3}{3}$

$-\ldots\ldots$ **(Patna 37, Utkal 50)**

7. If $x^2y = 2x - y$ and $x < 1$, show that

$$y + \frac{y^3}{3} + \frac{y^5}{5} + \ldots\ldots = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)$$

AGRA UNIVERSITY PAPERS

Fully solved 1948—56.

ALGEBRA

1948

1. (a) Show that $x=2$ is a root of the equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

and solve it completely.

(b) Express as a sum of partial fractions

$$\frac{x^3}{(x+2)^2(x^2+2)}$$

See question 5 (f) page 303 and question 10 (b) page 28

2. Test the convergency of the series

(i) $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots\ldots$ to ∞

(ii) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}$ $+\ldots\ldots\ldots$ to ∞

See question 6 page 190 and question 1 page 203.

3. Prove that

$$(1+x)^{1+x}(1-x)^{1-x} > 1, \text{ if } x < 1,$$

hence deduce that

$$a^a \cdot b^b > \left[\frac{a+b}{2}\right]^{a+b}$$

See § 14 page 114.

4. (a) Prove that the difference in value between a simple continued fraction and its nth convergent lies

between

$$\frac{1}{q_n q_{n+1}} \text{ and } \frac{1}{q_n(q_{n+1}+q_n)}$$

(b) In the continued fraction

$$k+\frac{1}{k+}\,\frac{1}{k+}\,\frac{1}{k+}\cdots\cdots$$

show that $p_n = q_{n+1}$

See Prop 5·12 page 62 and question 7 (b) page 79.

1949

1. (a) Resolve into partial fractions :

$$\frac{x^4-3x^3-3x^2+10}{(x-1)^2(x-3)}$$

(b) If n is a positive integer and $x<1$, show that

$$\frac{1-x^{n+1}}{n+1} < \frac{1-x^n}{n}$$

See question 1 (b) page 22 and question 7 page 136.

2. (a) If $\frac{p_n}{q_n}$ be the nth convergent to a continued fraction, then prove that

$$p_n\, q_{n-1} - q_n\, p_{n-1} = (-1)^n$$

Convert $\sqrt{(10)}$ into a continued fraction.

(b) Show that

$$\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\cdots\cdots$$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\cdots\cdots$$

See §5 page 55, question 2 (c) P. 72 and Ex. 3 P. 338.

3. Test the convergency or divergency of the series :—

(a) $\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\cdots\cdots\infty$

(b) $1+\frac{2x}{2\,!}+\frac{3^2\,x^2}{3\,!}+\frac{4^3\,x^3}{4\,!}+\frac{5^4\,x^4}{5\,!}+\cdots\cdots\infty$

See question 2 page 221 and question 2 page 204.

4. Express

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$$

as a product of two determinants; and hence or otherwise, find its value.

See question 4 (i) page 295.

1950

1. (a) Resolve into partial fractions

$$\frac{2x^2-11x+5}{(x-3)(x^2+2x-5)}.$$

(b) Find the sum to infinity of the series

$$1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\dots\dots\dots$$

See question 2 (b) page 22 and Ex. 1. P. 327

2. (a) Find the value of

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

(b) Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}=2abc(a+b+c)^3$$

See Ex. 6 page 260 and question 3 (i) page 283.

3. Find whether the following series are convergent or divergent :—

(a) $2/1^p+3/2^p+4/3^p+5/4^p+\dots\dots\dots$

(b) $1+a+\frac{a(a+1)}{1.2}+\frac{a(a+1)(a+2)}{1.2.3}+\dots\dots$

See Q. 9, page 179 and Q. 6 (a), page 208.

4. (a) a and b are positive and $a > b$

prove that $a^n - b^n > n(a-b).(ab)^{(n-1)/2}$

(b) In the continued fraction

$$\frac{1}{a+}\;\frac{1}{a+}\;\frac{1}{a+}\;\frac{1}{a+}+\dots\dots$$

show that

(i) $p_n^2+p_{n+1}^2=p_{n-1}p_{n+1}+p_np_{n+2}$

(ii) $p_n=q_{n-1}$

Convert 0·37 into a continued fraction.

See question 8 P. 137 and question 7 page 79 and Ex. 1 (b) page 48.

1951

1 (a) Resolve into partial fractions :—

$$\frac{x^4}{(x-1)^4(x+1)}$$

(b) If $x<1$, find the sum of the series

$$\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\frac{4}{5}x^5+\dots\dots\text{to infinity}$$

See Q. 8 (b) P. 27, and Ex. 4 P. 338.

2. (a) In the continued fraction

$$\frac{1}{a+}\;\frac{1}{b+}\;\frac{1}{a+}\;\frac{1}{b+}\dots\dots\dots$$

show that

$p_{n+2}-(ab+2)p_n+p_{n-2}=0$ and $q_{n+2}-(ab+2)q_n+q_{n-2}=0$

(b) If a, b, c are in H. P. and $n>1$, show that

$$a^n+c^n>2b^n.$$

See Q. 8 (b) page 81 and Q. 3 (c) page 130

8. Test the convergency or divergency of the series :—

(i) $1+\frac{2^2}{2!}+\frac{3^3}{3!}+\frac{4^4}{4!}+\ldots\ldots$

(ii) $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots\ldots$

See Q. 15 P. 193 and Q. 8 P. 209.

4 (a) Show that

$$\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = 2(b-c)(c-a)(a-b)\times(y-z)(z-x)(x-y)$$

(b) Solve the equation

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

See Q. 4 (i) page 295 and Q. 5 (c) P. 302.

1 9 5 2

1. (a). Express as a sum of partial fractions

$$\frac{9x^3-24x^2+48x}{(x-2)^4(x+1)}$$

(b) Show that

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} > \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

where a, b, c are positive quantities.

See Ex. 2 (a) P. 15 and question 3 (k) P. 133.

7. (a) Show that the infinite series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots+\frac{1}{n^p}+\ldots\ldots$$

is convergent if and only if $p>1$.

(b) Discuss the convergency of

$$\Sigma \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots2(n+1)}x^n.$$

See § 9 P. 170 and question 1 (b) P. 201, $x<1$ (c) $x \geqslant$ D.

3. (a) Prove that the difference in value between a simple continued fraction and its *n*th convergent lies between

$$\frac{1}{q_n\,q_{n+1}} \text{ and } \frac{1}{q_n(q_{n+1}+q}\ )$$

where the symbols have the usual meanings.

(b) Express 763/396 as a continued fraction, and hence find the least value of x and y which satify

$$396x-763y=12.$$

See § 5·12 Page 62 and question 4 (c) Page 75.

9. Evaluate the determinants .—

(1) $\begin{vmatrix} 2bc-a^2, & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$ and (2) $\begin{vmatrix} x & p & p & p \\ p & x & p & p \\ p & p & x & p \\ p & p & p & x \end{vmatrix}$

See question 4 (g) Page 293 and question 2 (n) Page 271.

1 9 5 3

1. (a) When $x \angle 1$ find the sum of the infinite series

$$\frac{1}{(1-x)(1-x^3)}+\frac{x^2}{(1-x^3)(1-x^5)}+$$

$$\frac{x^4}{(1-x^5)(1-x^7)}+\ldots\ldots\ldots$$

(b) If $x<1$, expand $[\log_e(1+x)]^2$ in ascending powers of x.

See Q. 4 (a) P. 34 and § 3 P. 336.

2. (a) If S be the sum of n +ive unequal quantities. $a, b, c, \ldots\ldots$ show that

$$\frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}\ldots\ldots>\frac{n^2}{n-1}$$

(b) If a, b, c are in H. P. and $n>1$, show that

$$a^n+c^n>2b^n.$$

(a) See Q. 6 P. 135. (b) See Q. 3 part (c) P. 130.

3. (a) If $\frac{p_n}{q_n}$ is the nth convergent to a continued fraction whose value is x, show that the difference between x and $\frac{p_n}{q_n}$ is less than $\frac{1}{q_n{}^2}$ and greater than $\frac{1}{2q^2{}_{n+1}}$

(b) Show that if a_n is the quotient corresponding to $\frac{p_n}{q_n}$

(i) $$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\,\ldots\ldots\frac{1}{a_2+}\,\frac{1}{a_1}$$

(ii) $$\frac{q_n}{q_{n-1}}+a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\,\ldots\ldots\frac{1}{a_3+}\,\frac{1}{a_2}$$

(a) See § 5·12 P. 63. (b) See Ex. 2 P. 54.

4. (a) Evaluate $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 227 & 203 \\ 221 & 201 & 201 \end{vmatrix}$

(b) Prove that
$$\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix}$$

$$=a_1a_2a_3a_4\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\right)$$

(a) See Ex. 6 P. 260. (b) see Part a P. 273,

1. (a) If ω is a cube root of unity, prove that $a+b\omega+c\omega^2$ is a factor of
$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Hence evaluate the determinant.

(b) Find the value of
$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

See Q. 1 (k) P. 264.

2. (a) Find the least value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ for positive values of x, y, z which satisfy the condition $x+y+z=c$.

(b) By means of the expansion of e^x and $\log(1+x)$, prove that when n is large,

$$\left(1+\frac{1}{n}\right)^n=e\left(1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{7}{16n^3}+\ldots\ldots\right)$$

Show that e is given approximately by $2e=(1{\cdot}1)^{10}+(0{\cdot}9)^{-10}$ with an error of about 0 46 per cent.

See page 131 Part e and Ex. 10 page 343.

3. (a) Show that the series $\Sigma\, n^{-p}$ is convergent if $P>1$ and divergent if $P\leqslant 1$.

(b) Test the convergence or divergence of the series

$$\frac{1}{1+x}+\frac{x}{1+x^2}+\frac{x^2}{1+x^3}+\ldots+\frac{x^{n-1}}{1+x^n}+\ldots\ldots(x>0)$$

(a) See § 9 P. 70.

(b) See P. 183 Q. 23 and 24.

4. (a) If p_n/q_n is the nth convergent of the continued fraction

$$\frac{1}{a+}\;\frac{1}{b+}\;\frac{1}{a+}\;\frac{1}{b+}\cdots\cdots\cdots$$

Prove that $p_n-(ab+2)p_{n-2}-p_{n-4}=0$

$$q_n-(ab+2)q_{n-2}-q_{n-4}=0.$$

(b) Express $\frac{157}{68}$ as a simple continued fraction with an odd number of quotients and find the least values of x and y which satisfy the equation $68x-157y=1$.

(a) See Q. 8 part (c) P. 81.

(b) See Q. 4 part (d) P. 75.

1 9 5 5

100% of the questions set were from this book.

1. (a) If ω is one of the cube roots of unity, show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix}$$

(b) Prove the identity

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} =(y-z)(z-x)(x-y)(xy+yz+zx).$$

(a) Part (d) P. 291, (b) Q. 3 (a) P. 278.

2. **Test the convergence of the following series :—**

(i) $1+\frac{2^2}{3^2}+\frac{2^2 . 4^2}{3^2 . 5^2}+\frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2}+\ldots\ldots \infty$

(ii) $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots\ldots \infty$

(a) Ex. 3 P. 218. (b) Q. 6 (a) P. 190.

3 (a) If a, b, c are unequal and positive quantities, prove that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}<\tfrac{1}{2}(a+b+c).$$

(b) Show that

$$1+\frac{2^3}{2\,!}+\frac{3^3}{3\,!}+\frac{4^3}{4\,!}\ldots\ldots\ldots=5e.$$

(a) Part iv P. 127, (b) Ex. 1. P. 327.

4. (a) If p_n/q_n be the nth convergent of a continued fraction, show that

$$p_n q_{n-1}-p_{n-1}\, q_n=(-1)^n.$$

(b) Express $\sqrt{7}$ as a simple continued fraction.

(a) § 4. P. 51, (b) Q. 2 (b) P. 72.

1 9 5 6

60% of the paper was set from this book.

1. If a, b, c, d......are p positive integers, whose sum is equal to n, show that

(a) the least value of $a\,!\, b\,!\, c\,!\, d\,!$......is

$$(q\,!)^{p-r}[(q+1)\,!^r],$$

and (b) the greatest value of the product $abcd$......is

$$q^{p-r}\,(q+1)^r,$$

where $n=pq+r$.

We have proved in Q. 21 P. 146 that if $n > r$ then

$$n!\ n! < r!\ (2n-r)! \ldots\ldots(1)$$

In a similiar manner we can prove that

$$n!\ (n+1)! < r!\ (2n+1-r)! \ldots\ldots(2)$$

From (1) we find that the product of two equal factorials whose sum is constant is least when they are replaced by equal factorials. From (2) we find that product of two unequal factorials whose sum is constant and odd will be least when they are replaced by two factorials which differ by unity.

We have to find the least value of

$$a!\ b!\ c!\ d! \ldots\ldots p \text{ factorials} \ldots\ldots\ldots(3)$$

where $a+b+c+d+\ldots\ldots\ldots=n$.

Now the value of (3) can be reduced by replacing any two unequal factorials by equal ones and it will be least when all the factorials are equal. This case is not possible because if $a=b=c\ldots\ldots p$ numbers then their sum n should be some multiple of p, but we are given that $n=pq+r$. i.e. n is not an exact multiple of p.

Hence we take the second choice i. e. any two of the factorials (3) should not differ by more than unity.

Now $n=pq+r=(p-r)q+r(q+1)$ i. e. we should choose $(p-r)$ factorials each equal to $q!$ and the remaining r factorials each equal to $(q+1)!$.

Hence the least value of (3) is

$$[q!\ q! \ldots\ldots (p-r) \text{ times}]\ [(q+1)!\ (q+1)! \ldots\ldots r \text{ times}]$$
$$=(q!)^{p-r}\ [(q+1)!]^{r}$$

(b) We have proved in § 5. P. 96 that the greatest value of the product $abcd\ldots\ldots\ldots p$ numbers whose sum is constant say n is max. when all the quantities are equal, but as in part (a) this is not possible since n is not an exact multiple of p. Again $n=pq+r=(p-r)q+r(q+1)$.

Hence the value will be greatest if we choose $p-r$ numbers each equal to q and the rest r numbers each equal to $q+1$ so that their sum is the given number n and value of the product in this case becomes

$$[q \cdot q \cdot q \ldots\ldots (p-r) \text{ times}][(q+1).\ (q+1)\ldots\ldots r \text{ times}]$$
$$=q^{p-r}(q+1)^r.$$

2. Investigate the convergency and divergency of the following series :—

$$\text{(i)}\quad \frac{(1+a)(1+b)}{1.2.3}+\frac{(2+a)(2+b)}{2.3.4}+\ldots\ldots$$
$$\frac{(n+a)(n+b)}{n(n+1)(n+2)}+\ldots\ldots\ldots$$

$$\text{(ii)}\quad 1+\frac{1}{2}x+\frac{2\,!}{3^2}x^2+\frac{3\,!}{4^3}x^3+\ldots\ldots\ldots$$

Part C. P. 224, Q. 3 P. 206.

3. In the case of a simple continued fraction, show that

$$p_n q_{n-1}-q_n p_{n-1}=(-1)^n,$$

where p_n and q_n have their usual meanings.

Expressing

$$\frac{x^3+x+1}{x^2+1}$$

as a simple continued fraction, find two polynomials A and B such that

$$A(x^3+x+1)-B(x^2+1)=\pm 1$$

(a) § 5 Prop. I.P. 55. (a) Q. 4 (a) P. 74.

4. (a) Evaluate :—

$$\begin{vmatrix} 3 & 7 & 9 & 6 \\ 8 & 4 & 5 & 8 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$$

(b) Prove that

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

$$=(b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2).$$

(a) Part (j) P. 263. (b) **Part k P. 284.**

5 (a) Give a rigorous proof of the Bionomial theorem for any index.

§ 5. P. 317.

RAJPUTANA UNIVERSITY PAPERS

Fully Solved 1948–56

ALGEBRA

1948

Same as Agra University 1948.

1949

1. (a) Show that; if a, b; c, d be four positive unequal quantities, if $s=a+b+c+d$, then

$$(s-a)(s-b)(s-c)(s-d)>81abcd.$$

(b) In the continued fraction

$$a+\frac{1}{b}+\frac{1}{a}+\frac{1}{b}+\frac{1}{a}+\frac{1}{b}+\frac{1}{a}+\ldots\ldots$$

odd order quotients being a, prove that $p_n=q_{n+1}$ and $aq_{2n}=bp_{2n-1}$.

See Q. 1 (z) P. 128 and Q. 8 P. 80.

2. Test the convergency of the series—

(i) $1+\frac{2p}{2!}+\frac{3p}{3!}+\frac{4p}{4!}+\ldots\ldots$

(ii) $x^2(\log 2)^a+x^3(\log 3)^a+x^4(\log 4)^a+\ldots\ldots$

See Q. 2 P. 189 and Q. 9 P. 211.

3. (a) Break $\frac{x-2}{(x+2)(x-1)^2}$ into partial fractions and show that the coefficient of x^n in the Binomial expansion is

$$\frac{4}{9}\left(-\frac{1}{2}\right)^{n+1}-\frac{1}{9}(3n+7)$$

(b) Identifying as a Binomial expansion, show that

$$\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots\ldots=0{\cdot}4 \text{ nearly.}$$

See Q. 13 part (j) P. 34 and Ex. 1 P. 320.

4. (a) Evaluate

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}$$

(b) Show that

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

is divisible by λ^3 and find the other factor.

See Q 3 (n) P. 28 and Q. 2 (t) P. 274.

1 9 5 0

1. (a) Show that if $a>b>c$, then

$$\left(\frac{a+c}{a-c}\right)^a<\left(\frac{b+c}{b-c}\right)^b$$

(b) Show that

$$\frac{3.7.11\ldots\ldots(4n-1)}{5.9.13\ldots\ldots(4n+1)}<\sqrt{\left(\frac{3}{4n+3}\right)}$$

See Ex. 1 P. 113 and Q. 10 P. 138.

2. Assuming the Binomial theorem for +ive integral index, prove it for any rational index.

If $\log_e(1-x+x^2)$ be expanded in ascending powers of x in the form $a_1x+a_2x^2+a_3x^3+\ldots\ldots$ then will

$$a_3+a_6+a_9+\ldots\ldots=\frac{2}{3}\log_e 2.$$

See §5 P. 317 and Ex. 5 P. 339.

3. (a) Prove that the infinite series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots\dots$$

is always divergent except when p is positive and greater than 1.

(b) Test the convergency of the series

(i) $x+x^{1+\frac{1}{2}}+x^{1+\frac{1}{2}+\frac{1}{3}}+\dots\dots$

(ii) $\frac{1}{\sqrt{2}-1}+\frac{1}{\sqrt{3}-1}+\frac{1}{\sqrt{4}-1}+\dots\dots$

See § 9 P. 170 and Q. 3 (c) P. 224.

4. (a) Find the value of

$$\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\dots\dots\dots$$

(b) Shsw that

$$\begin{vmatrix} b^2+c^2+1 & c^2+1 & b^2+1 & b+c \\ c^2+1 & c^2+a^2+1 & a^2+1 & c+a \\ b^2+1 & a^2+1 & a^2+b^2+1 & a+b \\ b+c & c+a & a+b & 3 \end{vmatrix}$$

$$=(bc+ca+ab)^2$$

Proceed as in Ex. 2 P. 69 and Q. 4 (q) P. 301.

1951

1. (a) If $x+y+z=1$, show that the least value of

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \text{ is } 9.$$

and that $(1-x)(1-y)(1-z)>8xyz$.

(b) In the continued fraction

$$\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\dots\dots$$

show that $p_n=q_{n-1}$.

See Q. 1 (l) P. 123, Q. 2 (e) P. 131 and Q. 7 (b) P. 79.

2. (a) 'An infinite series is convergent if each term is less than the preceding term.'

Modify the above statement, illustrating your answer by considering the series

$$1+2\,.\,\tfrac{4}{5}+3(\tfrac{4}{5})^2+4(\tfrac{4}{5})^3+5(\tfrac{4}{5})^4+6(\tfrac{4}{5})^5+\ldots\ldots$$

(b) Find whether the series

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\frac{5^5x^5}{5!}+\ldots\ldots$$

is convergent or divergent.

See § 11 P. 184 and Ex. P. 202.

3. (a) Resolve into partial fractions—

$$\frac{x-2}{(x+2)(x-1)}$$

and find the coefficient of x^n in the expansion.

(b) Shew that

$$\begin{aligned}\log_e \sqrt{(12)}=1&+(\tfrac{1}{2}+\tfrac{1}{3})\tfrac{1}{4}+(\tfrac{1}{4}+\tfrac{1}{5})(1/4^2)\\&+(\tfrac{1}{6}+\tfrac{1}{7})(1/4^3)+(\tfrac{1}{8}+\tfrac{1}{9})(1/4^4)+\ldots\ldots\end{aligned}$$

Very simple and see Ex. 7 page 348.

4. (a) Prove that

$$\begin{vmatrix}(b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix}=2abc(a+b+c)^3$$

(b) Express

$$\begin{vmatrix}2bc-a^2 & c^2 & b^2\\ a^2 & 2ac-b^2 & a^2\\ b^2 & a^2 & 2ab-c^2\end{vmatrix}$$

as the product of two determinants and find its value.

Q. 3 (i) page 283 and Q. 4 (g) page 293.

1952

1. (a) If any two of a, b, c are together greater than the third, show that

$$(a+b+c)^3 > 27\,(b+c-a)(c+a-b)(a+b-c).$$

(b) If $\frac{p_n}{q_n}$ is the nth convergent to the continued fraction

$$\frac{1}{a+}\;\frac{1}{b+}\;\frac{1}{a+}\;\frac{1}{b+}\;\frac{1}{a+}\cdots\cdots$$

shew that $q_{2n}=p_{2n+1}-b\cdot q_{2n-1}=\frac{a}{b}p_{2n}$.

See question 1 (g) P. 121 and question 8 P. 80.

2. (a) Show that the series

$$\frac{1}{1^2-x}+\frac{1}{2^2-x}+\frac{1}{3^2-x}+\cdots+\frac{1}{n^2-x}$$

is convergent for all values of x, except only when x is the square of an integer

(b) Test for the convergency of the series :—

$$\frac{a+x}{1\,!}+\frac{(a+2x)^2}{2\,!}+\cdots\cdots\frac{(a+nx)^n}{n\,!}+\cdots\cdots$$

See question 8 P. 209.

3. (a) Resolve into partial fractions

$$\frac{x^2+3}{(x^2+2)(x-1)^2}$$

and find the coefficient of x^n in the expansion.

Sum the series to infinity

$$2+\frac{5}{2\,!\,.\,3}+\frac{5\,.\,7}{3\,!\,.\,3^2}+\frac{5\,.\,7\,.\,9}{4\,!\,.\,3^3}+\cdots\cdots$$

Proceed as in question 12 (c) P. 30 and Ex 3. P 221.

4. (a) Prove that

$$\begin{vmatrix} 0 & c & b & a \\ c & 0 & a & b \\ b & a & 0 & c \\ a & b & c & 0 \end{vmatrix}$$

$=(a+b+c)(b+c-a)(c+a-b)(c+a-b).$

(b) Show that

$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix}=(ax-by+cz)^2.$$

See Q. 3 (m) P. 287 and Q. 2 (q) P. 272.

1 9 5 3

1. (a) If n be a positive integer greater than 2, show that $2^n > 1+n\sqrt{2^{n-1}}$.

(b) A lunar month consists of 29·53 days and a tropical year of 365·24 days. Show with the help of continued fraction that there are nearly 99 lunar months in 8 tropical years.

Q. 1 (ii) page 126 & question 5 (e) page 77.

2. (a) Resolve into partial fractions :—

$$\frac{x^4-5x^3+10x^2-8x-1}{(x-1)^3(x-2)}$$

(b) Test the convergency of the series :—

$$1+\frac{2x}{2!}+\frac{3^2.x^2}{3!}+\frac{4^3.x^3}{4!}+\ldots\ldots$$

Q. 1 (a) page 21 and question 2 page 204.

3. (a) Show that

$$\log_e \sqrt{(12)}=1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}$$

$$+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\left(\frac{1}{8}+\frac{1}{9}\right)\frac{1}{4^4}\ldots\ldots\text{to } \infty$$

(b) F[illegible] the determinant :—

$$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$$

Ex. 7 page 340 & question 1 (g) page 263

1954

1. (a) Show that if all the factors are positive.

$$abc>(b+c-a)(c+a-b)(a+b-c)$$

(b) If $x=\frac{a}{1}+\frac{b}{1}+\frac{a}{1}+\ldots\ \ldots$

and

$$y=\frac{b}{1}+\frac{a}{1}+\frac{b}{1}+\ldots\ldots\ldots$$

Show that $x-y=a-b$.

Q.. 1 (g) page 121 and proceed as in Ex. 1. page 69.

2. (a) Express as a sum of partial fractions

$$\frac{x^3}{(x+2)^2(x^2+2)}$$

(b) Find whether the series

$$x+x^{1+\frac{1}{2}}+x^{1+\frac{1}{2}+\frac{1}{3}}+x^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+}+\ldots\ldots$$

is convergent or divergent.

simple [illegible]ion 3 (c) page 224.

3. (a) If $y=\frac{1}{3}+\frac{1\,.\,3}{3\,.\,6}+\frac{1\,.\,3\,.\,5}{3\,.\,6\,.\,9}+\ldots\ldots\ldots$

prove that $y^2+2y-2=0$

(b) Express

$$\begin{vmatrix} b^2+c^2 & [illegible] & [illegible] \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix}$$

as the square of a determinant and find its value.

Ex. 2 page 321 & see question 4 (n) page 299.

1955.

100% of the questions were from this book.

1. (a) Prove that

$$x^y y^x < \left(\frac{x+y}{2}\right)^{x+y}$$

where x and y are positive quantities.

(b) Resolve the following into partial fractions :—

$$\frac{7x^2-12x+11}{(x-1)^2(x^2+1)}$$

Expand the above in ascending powers of x, ($|x|$ being less than 1) and find the coefficient of x^r when r is an even integer.

(a) § 16 page 115. See parts (d) and (e) page 31.

2. Prove that the infinite series Σu_n of positive terms is convergent or divergent according as after some fixed term

$$u_n^{1/n} \leqslant k < 1 \quad \text{or} \quad u_n^{1/n} \geqslant 1.$$

Test the following series for convergence or divergence :—

(i) $\Sigma\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-n}$

(ii) $\Sigma\frac{(a+nx)^n}{n!}$

§ 12 page 147, Q. 16 page 193, Q. 8 page 209.

3. (a) [illegible] if [illegible] be the nth convergent to the continued fraction

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots\cdots$$

then

$$q_{2n}=p_{2n+1},\ bq_{2n+1}=ap_{2n}+abp_{2n+1} \text{ and } q_{2n-1}=\frac{a}{b}p_{2n}$$

(b) Shew that the infinite series

$$1+\frac{1}{4}+\frac{1.4}{4.8}+\frac{1.4.7}{4.8.12}+\frac{1.4.7.10}{4.8.12.16}+\cdots\cdots$$

and

$$1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\frac{2.5.8.11}{6.12.18.24}+\cdots\cdots$$

are equal.

Q. 8 page 80, Q. 5 page 323.

4. (a) Prove that the product of two determinants of the same order is itself a determinant of the same order.

(b) Prove that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$=(a^3+b^3+c^3-3abc)^2$$

(a) § 18 P. 255, (b) Part (g) P. 293.

1 9 5 6

88% of the paper was set from this book.

1. (a) Find the value of

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 2 & 2+x & 2 & 2 \\ 3 & 3 & 3+x & 3 \\ 4 & 4 & 4 & 4+x \end{vmatrix}$$

(b) Shew that

$$\begin{vmatrix} x^2-yz & y^2-zx & z^2-xy \\ z^2-xy & x^2-yz & y^2-zx \\ y^2-zx & z^2-xy & x^2-yz \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2.$$

Part (o) P. 271. Part (b) P. 289.

2. (a) If s be the sum of n positive unequal quantities $a, b, c, \ldots$, prove that

$$\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+\ldots>\frac{n^2}{n-1}$$

(b) Prove that the error in taking p_n/q_n instead of the whole continued fraction is less than

$$\frac{1}{a_{n+1}q_n^2}.$$

Hence prove that $\frac{355}{113}$ is a very near approximation to 3·14159.

Q. 6 page 135. § 5·12 result D page 65, Ex. 2 page 66

3. (a) Resolve into partial fractions :—

$$\frac{x^2-x+1}{-1)^2(x-2)(x^2+1)}$$

(b) If n people seat themselves at a round table, what is the chance that two named individuals be neighbours ?

Q. 4. page 23 (Chapter not in the book.)

4. (a) If

$$a_n=\left(1+\frac{x}{n}\right)e^{-x/n}$$

find whether the series, whose nth term is a_{n-1}, is convergent or divergent.

[illegible]

(b) Fi[illegible]ing series is convergent or divergent :—

$$x^2+\frac{2^2}{3\,.\,4}x^4+\frac{2^2\,.\,4^2}{3\,.\,4\,.\,5\,.\,6}x^6+\frac{2^2\,.\,4^2\,.\,6^2}{3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8}x^8+\ldots\ldots$$

Q. 25 page 184. Q. 5 page 207.